$$\frac{\text{Conjecture 1}: (\text{Horel}) \times \text{simplicial sheaf}, \text{ then}}{\mathcal{T}_{o}^{A^{1}}(\mathbf{x}) \text{ is } A^{1} - \text{invariant}}.$$

$$\frac{\text{Conjecture 2}: \times \text{ smooth scheme over } \kappa, \text{ the natural}}{g(\mathbf{x}) \longrightarrow \mathcal{T}_{o}^{A^{1}}(\mathbf{x}) \text{ is an iso}}.$$

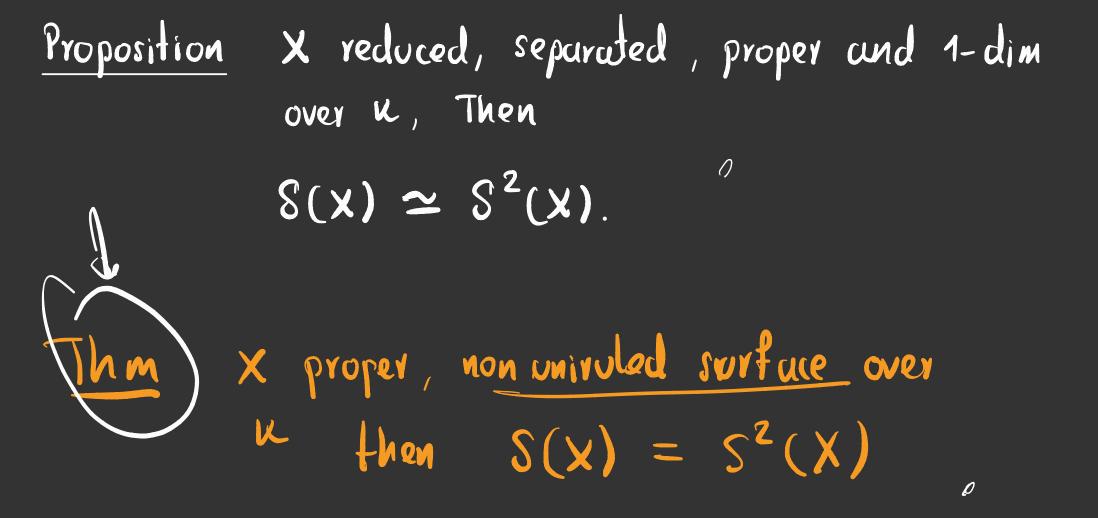
Goal: Conjectures 1 and 2 hold for non-uniruled surfaces over K. ?

Lemma: Let F sheat over Sm_{k} , then $S(F) = S^{2}(F)$ \iff \forall \mathcal{U} smooth Henselian local, if $t_{1}, t_{2} \in F(\mathcal{U})$ are ghost - homotopic then they are A^{1} - cheeln homotopic.

+ reduction to 1-dimensional schemes.

<u>Def</u>: X scheme over K, $f: Y \longrightarrow X$ is a \mathbb{P}^{1} -fibration if f is smooth, proper and $\forall x \in X$, $f^{-1}(x)$ is a radional curve.

Lemma: E, B varieties over K IT: E -> B smooth, projective over K YbeB, IT⁻¹(b) = Eb isomorphic to IP¹_b Then IT is an étale locally trivial fiber bundle.



<u>Pef</u>: A K-variety of dimension n is <u>ruled</u> (resp. uninuled) if I a K-scheme Y of dimension n-1 and a rational map: Y: Y X P¹ ---> X which is <u>birradional</u> (resp. dominant).

non-uniruled = not uniruled. (=) $\forall \forall x$ -scheme of dim n-1 and $\forall \forall x P^1 - - \Rightarrow X$ rational then \forall cannot be dominant. Proposition (Kollar, Rational curves on alg varieties IV. 1.3)

X variety over K, then t.f. a.e:

(1) X is univoled over K
(2) J Z variety over K + dominant P¹×Z--->X and ZEZ s.t the induced map P¹₂ --->X is non constant

(3) J K-varieties Z and U over K + a K-morphism

g: $U \longrightarrow Z$ with fibers rational curves and p: $U \longrightarrow X$ dominant s.t for some $Z \in Z$ the induced map $g^{-1}(Z) \longrightarrow X$ is non-const. Prop: X reduced scheme of dim 1 over K separated proper

Then $S(X) = S^2(X)$.

Proof. Suppose that X is smooth over K. Let U
smooth Henseliun local over K
$$\left(\begin{array}{c} \mathrm{Idea} \\ \mathrm{S}(\mathrm{X})(\mathrm{U}) \simeq \mathrm{S}^{2}(\mathrm{X})(\mathrm{U}) \right)$$

We have that the fibers of X are 1-dim smooth
varieties over K (X proper) so we can
write

I is finite.

If Ci is not rational over K (K alg closure of K) in particular Ci is not birational to $\mathbb{P}_{\overline{k}}^{T}$ then the genus Ci is ≥ 1 (By remark on the dussification of smooth curves over a field -> Lemma 2.1.11 -> Asoll-Movel)

Thus,

$$C_{i} \longrightarrow S(C_{i}) \longrightarrow J_{0} C_{i} \longrightarrow C_{i}$$

$$Al^{1} C_{i} \longrightarrow C_{i}$$

$$Al^{1} rigidily$$

$$2.2.5$$

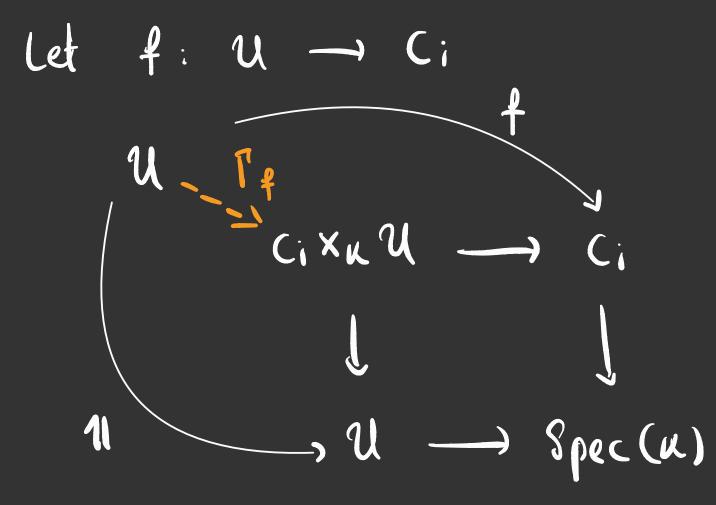
$$Al^{1} rigidily$$

$$Al^{1} rigidily$$

 $\begin{array}{c} C_{i}(u) \rightarrow U_{i}C_{i}J_{A1} \\ C_{i} \rightarrow J_{0}(c_{i}) \end{array}$

As a consequence: $S(C_i) \simeq C_i$ so that $S(C_i)(u) \simeq S^2(C_i)(u)$





so that f factors as:

 \mathcal{I}_{+} $\mathcal{U} \xrightarrow{\mathcal{I}_{+}} C_{i \times \mathcal{U}} \xrightarrow{\mathcal{I}_{+}} C_{i}$ 4

If $Cix_{n} \mathcal{U} \longrightarrow \mathcal{U}$ has no sections then $S(Ci)(\mathcal{U}) = \emptyset$ $S^{2}(Ci)(\mathcal{U}) = \emptyset$

Otherwise, then CixkU -> U Smooth, proper st the fiber over any point $u \in U$ is $\cong P_{i}^{1}$ Then by previous Lemma, Cix, U -) U is an étale-locally trivial R'-bundle and U Heaselian local then it is a trivial IP1- fiber bundle Then $S(C_i)(U) = *$

$$S^{Z}(C_{i})(\mathcal{U}) = *$$

If X is not smooth: U smooth irred over K, and let h: $U \times AI^{1} \rightarrow X$ any $AI^{1} - htpy of$ U in X ($AI^{1} - ghost homotopy$) then h factors by the normalization:

$$u \ge Al^{1} \xrightarrow{h} X \qquad dim(x) = 1$$

 $u \ge Al^{1} \xrightarrow{h} X \qquad dim(x) = 1$
 $u \ge X^{1} \xrightarrow{h} x^{1} \qquad \text{then } X^{1} \xrightarrow{rs}$
 $regular$

so that if the image of h is dense in X ~) use the smooth case. If not the image is a single point. Theorem: X proper, non-uniruled surface over k then $S(x) \simeq S^2(X)$. Proof: (Idea of the proof): X is non-uniruled then by the equivalence we studied:

VZ variety over V and V IP'x Z --->X then (1) P¹xZ --->X is not dominant 01 (2) $\forall z \in Z$ $P_z^1 - - - > X$ is constant. For any U smooth Henselian local scheme: Goal: $S(X)(U) \xrightarrow{\sim} S(X)(UX AI)$

If dim (U) = 0 this follows from "Asok-Nord theorem".

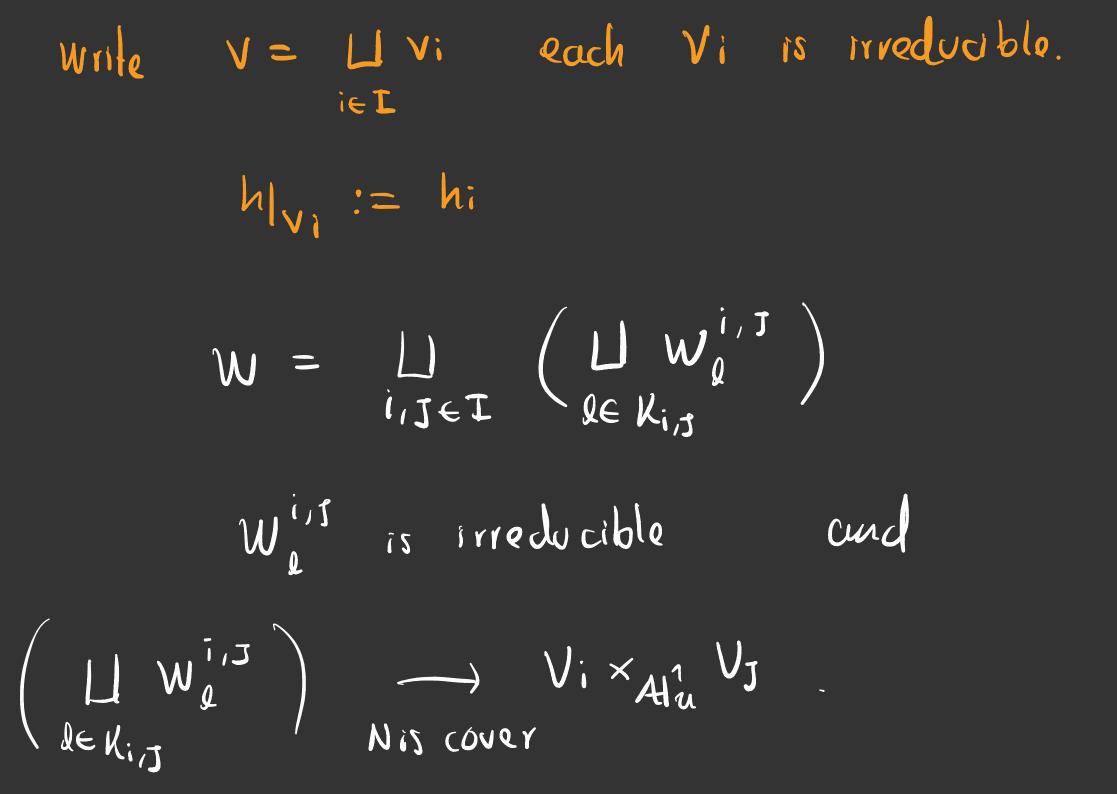
Suppose that $dim(y) \ge 1$. Take

$$H := (V \rightarrow Al'_{u}, W \rightarrow V \times V, h, h'')$$

here
$$h: \vee \longrightarrow X$$

an
$$A^{1}$$
-ghost htpy of U in X connecting
tates: $U \rightarrow X$. This A^{1} -ghost htpy determines

an Al'-htpy h: Al'n -> S(X), the idea is to show that: $(\mathcal{H} \times \mathcal{A}^1 \longrightarrow \mathcal{X})$ either R lifts to an hantpy of U in X the given Al'-ghost htpy factors through some 1-dim, reduced closed subscheme of X.



$$W_{Q}^{i,J} \longrightarrow V_{I} \times_{Al'_{u}} V_{J} \longrightarrow V_{i} \xrightarrow{h_{i}} \times$$

restrictions of h^{w} to $W_{Q}^{i,J}$ are either
tant Al^{2} - chain homotopies of

they factor through 1-dim reduced, closed subscheme of X (X is non-univuled!).

The

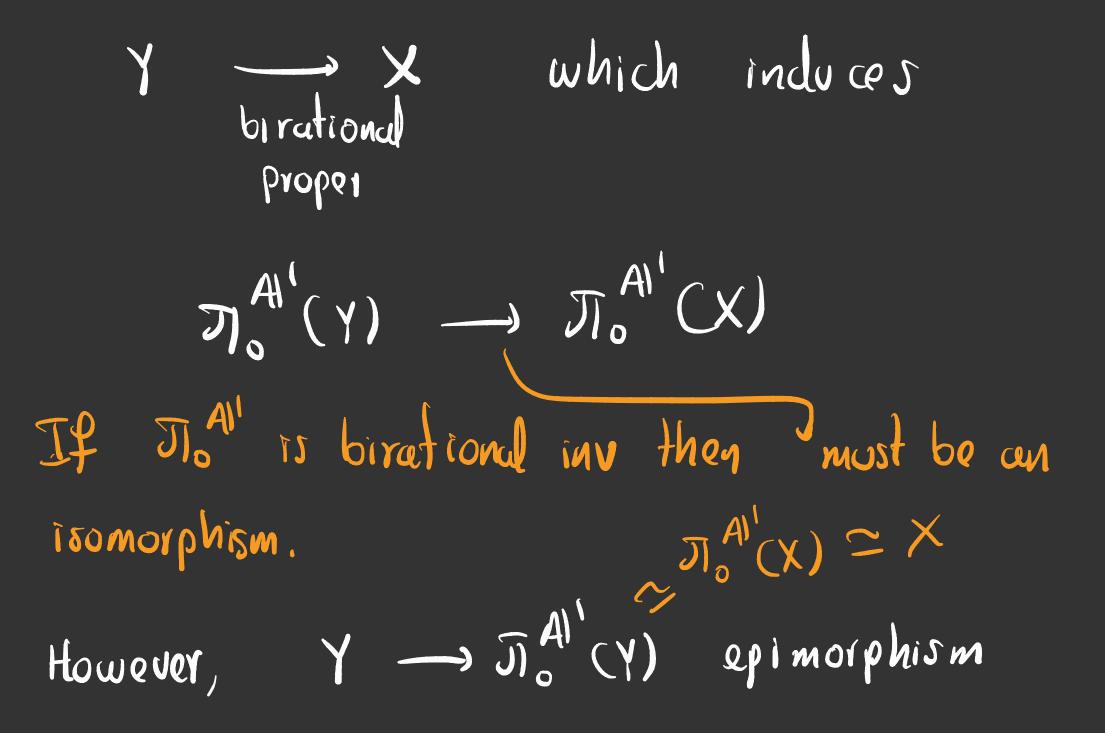
COAS

Corollary: Conj 1 and 2 hold for non-uniruled surfaces over a field.

Remark: They do not hold for ruled surfaces [Al¹ - connected components of ruled surfaces].

To is not birational invariant: XE SMK s.t X 15 All-rigid. \$2 X is simplifially fibrant, by 3.19 _ MV (X 15 Al'-local (=) is Al'-inv) So that if X is Al'-rigid : Vue Smr $X(\mathcal{U}) \xrightarrow{\sim} Hom_{H(\mathcal{K})}(\mathcal{U}, \mathbf{X})$

ro that
$$X \rightarrow J_{0}^{Al'}(X)$$
 is an
isomorphism of Nisn sheaves.
Example: X abelian variety of dim ≥ 2
By A.M. 2.1.10, X is Al'-rigid
so that $J_{0}^{Al'}(X) \rightrightarrows X$.
Next Blow up a point on X to get



Example of a scheme
$$\times$$
 s.t
Sing. (X) is not Al'-local.
X smooth projective variety over C s.t
(i) $S(X) \neq S^2(X)$
i) $S(X) \longrightarrow \overline{Ji}_{o}^{Al'}(X)$ is not a mono.
ii) $Sing_{x}(X)$ is not Al'-local
Sed. 4.1: const 4.5: (4)

