II. Morel's conjectures for non-uniruled surfaces

Conjecture 1: (Morel) $\mathcal{X}$ simplicial sheaf, then $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant.

Conjecture 2: X smooth scheme over k, the natural epimorphism

$$\mathcal{S}(X) \longrightarrow \pi_0^{\mathbb{A}^1}(X) \text{ is an iso.}$$

Goal: Conjectures 1 and 2 hold for non-uniruled surfaces over k.

Lemma*: Let $\mathcal{F}$ sheaf over Sm_k, then $S(\mathcal{F}) = S^2(\mathcal{F})$ ⤳ hold ($\dim \mathcal{U} > 0$)

Idea: $\Longleftrightarrow$ $\forall\, \mathcal{U}$ smooth Henselian local, if $t_1, t_2 \in \mathcal{F}(\mathcal{U})$ are ghost-homotopic then they are $\mathbb{A}^1$-chain homotopic.

+ reduction to 1-dimensional schemes.

Def: X scheme over k, $f: Y \longrightarrow X$ is a **$\mathbb{P}^1$-fibration** if f is smooth, proper and $\forall x \in X$, $f^{-1}(x)$ is a rational curve.

Lemma: E, B varieties over k

$\pi: E \longrightarrow B$ smooth, projective over k

$\forall b \in B$, $\pi^{-1}(b) = E_b$ isomorphic to $\mathbb{P}^1_b$

Then π is an étale locally trivial fiber bundle.

Proposition X reduced, separated, proper and 1-dim over k, Then

$$\mathcal{S}(X) \simeq S^2(X).$$

Thm X proper, non uniruled surface over k then $S(X) = S^2(X)$

<u>Def</u>: A k-variety of dimension n is <u>ruled</u> (resp. <u>uniruled</u>) if $\exists$ a k-scheme Y of dimension $n-1$ and a rational map:

$$\varphi : Y \times \mathbb{P}^1 \dashrightarrow X$$

which is <u>birrational</u> (resp <u>dominant</u>).

non-uniruled = not uniruled.

$\Leftrightarrow$ $\forall$ Y k-scheme of dim $n-1$ and $\varphi : Y \times \mathbb{P}^1 \dashrightarrow X$ rational then φ cannot be dominant.

Proposition (Kollar, Rational curves on alg varieties IV. 1.3)

X variety over k, then t.f.a.e:

(1) X is uniruled over k

(2) $\exists$ Z variety over k + dominant $\mathbb{P}^1 \times Z \dashrightarrow X$ and $z \in Z$ s.t the induced map $\mathbb{P}^1_z \dashrightarrow X$ is non constant

(3) $\exists$ k-varieties Z and $\mathcal{U}$ over k + a k-morphism

$g: U \to Z$ with fibers rational curves and

$p: U \dashrightarrow X$ dominant s.t for some $z \in Z$

the induced map $g^{-1}(z) \dashrightarrow X$ is non const.

Prop: X reduced ~~separated~~ proper scheme of dim 1 over k

Then $\mathcal{S}(X) = \mathcal{S}^2(X)$.

Proof. Suppose that X is smooth over k. Let $\mathcal{U}$ smooth Henselian local over k (Idea: $\mathcal{S}(X)(\mathcal{U}) \simeq \mathcal{S}^2(X)(\mathcal{U})$)

We have that the fibers of X are 1-dim smooth varieties over k (X proper) so we can write

$$X = \bigsqcup_{i \in I} C_i$$ C_i irreducible smooth projective curves

I is finite.

If C_i is not rational over $\bar{k}$ ($\bar{k}$ alg closure of k) in particular C_i is not birational to $\mathbb{P}^1_{\bar{k}}$ then the genus C_i is ≥ 1

(By remark on the classification of smooth curves over a field $\rightarrow$ Lemma 2.1.11 $\rightarrow$ Asok–Morel)

And C_i is $\mathbb{A}^1$-rigid

Fr X is $\underline{\mathbb{A}^1\text{-rigid}}$: $\forall\, U \in Sm_k$ $\quad X(U) \xrightarrow{\sim} X(U \times \mathbb{A}^1)$

induced.

$U \times \mathbb{A}^1 \longrightarrow U$.

Thus,

$$C_i \twoheadrightarrow S(C_i) \xrightarrow[\substack{\text{Asok-Morel}\\ 2.2.5}]{} \pi_0^{\mathbb{A}^1}(C_i) \xleftarrow[\substack{\mathbb{A}^1\text{-rigidity}\\ + \\ \text{AM } 2.1.9}]{\sim} C_i$$

$$\left[\begin{array}{ccc} & c_i(u) & \longrightarrow [u, c_i]_{\mathbb{A}^1} \\ c_i & \xrightarrow[\sim]{} & \pi_0^{\mathbb{A}^1}(c_i) \end{array}\right]$$

As a consequence:

$$S(c_i) \simeq c_i$$

so that

$$S(c_i)(u) \simeq S^2(c_i)(u)$$

- If C_i is rational over $\bar{k}$ then C_i is isomorphic to a smooth plane conic.

Let $f : U \longrightarrow C_i$

$$\begin{tikzcd}
U \arrow[dr, dashed, "\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathbb{1}"'] & & \\
& C_i \times_k U \arrow[r] \arrow[d] & C_i \arrow[d] \\
& U \arrow[r] & \mathrm{Spec}(k)
\end{tikzcd}$$

so that f factors as:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\Gamma_f"] \arrow[rr, bend right, "f"'] & C_i \times_{\mathcal{U}} \mathcal{U} \arrow[r] & C_i \end{tikzcd}$$

If $C_i \times_{\mathcal{U}} \mathcal{U} \longrightarrow \mathcal{U}$ has no sections then

$$S(C_i)(\mathcal{U}) = \emptyset$$
$$S^2(C_i)(\mathcal{U}) = \emptyset$$

Otherwise, then $C_i \times_K \mathcal{U} \to \mathcal{U}$ smooth, proper s.t the fiber over any point $u \in \mathcal{U}$ is $\cong \mathbb{P}^1_u$

Then by previous Lemma, $C_i \times_K \mathcal{U} \to \mathcal{U}$ is an étale-locally trivial $\mathbb{P}^1$-bundle and $\mathcal{U}$ Henselian local then it is a trivial $\mathbb{P}^1$- fiber bundle

Then

$$S(C_i)(\mathcal{U}) = *$$

$$\cong$$

$$S^2(C_i)(\mathcal{U}) = *.$$

If X is not smooth: $\mathcal{U}$ smooth irred over k, and let $h : \mathcal{U} \times \mathbb{A}^1 \to X$ any $\mathbb{A}^1$-htpy of $\mathcal{U}$ in X (or $\mathbb{A}^1$-ghost homotopy) then h factors by the normalization:

$$\begin{tikzcd} \mathcal{U} \times \mathbb{A}^1 \arrow[rr, "h"] \arrow[dr, dashed] & & X \\ & X' \arrow[ur] & \end{tikzcd}$$

$\dim(X) = 1$ then X' is regular

so that if the image of h is dense in X

$\rightsquigarrow$ use the smooth case.

If not the image is a single point. $\square$

<u>Theorem</u>: X proper, non-uniruled surface over $\mathbb{C}$

then $S(X) \simeq S^2(X)$.

<u>Proof</u>: <u>(Idea of the proof)</u>: X is proper non-uniruled

then by the equivalence we studied :

$\forall$ Z variety over k and $\forall$ $\mathbb{P}^1 \times Z \dashrightarrow X$

then

(1) $\mathbb{P}^1 \times Z \dashrightarrow X$ is <u>not</u> dominant

or

(2) $\forall z \in Z$ $\mathbb{P}^1_z \dashrightarrow X$ is constant.

<u>Goal</u>: For any U smooth Henselian local scheme:

$$S(X)(U) \xrightarrow{\sim} S(X)(U \times \mathbb{A}^1)$$

If $\dim(\mathcal{U}) = 0$ this follows from "Asok–Morel theorem".

Suppose that $\dim(\mathcal{U}) \geq 1$. Take

$$\mathcal{H} := \left(V \longrightarrow \mathbb{A}^1_{\mathcal{U}},\ W \longrightarrow V \times_{\mathbb{A}^1_{\mathcal{U}}} V,\ h,\ h^W \right)$$

here $h : V \longrightarrow X$.

an $\mathbb{A}^1$-ghost htpy of $\mathcal{U}$ in X connecting $t_1, t_2 : \mathcal{U} \longrightarrow X$. This $\mathbb{A}^1$-ghost htpy determines an $\mathbb{A}^1$-htpy $\tilde{h}: \mathbb{A}^1_{\mathcal{U}} \to S(X)$, the idea is to show that:

$(\mathcal{U} \times \mathbb{A}^1 \to X)$

either $\tilde{h}$ lifts to an hmtpy of $\mathcal{U}$ in X

or

the given $\mathbb{A}^1$-ghost htpy factors through some 1-dim, reduced closed subscheme of X.

Write $V = \coprod_{i \in I} V_i$ each V_i is irreducible.

$$h|_{V_i} := h_i$$

$$W = \coprod_{i,j \in I} \left(\coprod_{l \in K_{i,j}} W_l^{i,j} \right)$$

$W_l^{i,j}$ is irreducible and

$$\left(\coprod_{l \in K_{i,j}} W_l^{i,j} \right) \xrightarrow[\text{Nis cover}]{} V_i \times_{\mathbb{A}^1_u} V_j .$$

$$W_\ell^{i,j} \longrightarrow V_i \times_{\mathbb{A}^1_u} V_j \longrightarrow V_i \xrightarrow{h_i} X$$

The restrictions of h^W to $W_\ell^{i,j}$ are either constant $\mathbb{A}^1$-chain homotopies <u>or</u> they factor through some 1-dim reduced, closed subscheme of X (X is non-uniruled!).

$\boxed{\lrcorner}$

Corollary: Conj 1 and 2 hold for proper non-uniruled surfaces over a field.

Remark: They do not hold for ruled surfaces

[$\mathbb{A}^1$ - connected components of ruled surfaces].

$\pi_0^{\mathbb{A}^1}$ is not birational invariant:

$X \in Sm_k$ s.t X is $\mathbb{A}^1$-rigid.

X is simplicially fibrant, by 3.19 - MV §2

(X is $\mathbb{A}^1$-local $\Leftrightarrow$ is $\mathbb{A}^1$-inv) so that

if X is $\mathbb{A}^1$-rigid: $\forall\, U \in Sm_k$

$$X(U) \xrightarrow{\sim} \mathrm{Hom}_{H(k)}(U, X)$$

so that $\qquad X \longrightarrow \pi_0^{\mathbb{A}^1}(X)$ is an isomorphism of Nisn sheaves.

Example: X abelian variety of dim ≥ 2

By AM 2.1.10, X is $\mathbb{A}^1$-rigid so that $\quad \pi_0^{\mathbb{A}^1}(X) \xrightarrow{\sim} X$.

Next Blow up a point on X to get

$Y \xrightarrow[\text{proper}]{\text{birational}} X$ which induces

$$\pi_0^{\mathbb{A}^1}(Y) \longrightarrow \pi_0^{\mathbb{A}^1}(X)$$

If $\pi_0^{\mathbb{A}^1}$ is birational inv then must be an isomorphism.

$$\pi_0^{\mathbb{A}^1}(X) \simeq X$$

However, $Y \longrightarrow \pi_0^{\mathbb{A}^1}(Y)$ epimorphism $\simeq$

thus the Blow-up morphism:

$Y \xrightarrow[\text{birr + proper.}]{} X$ is an epimorphism

$\Rightarrow$ $Y \to X$ must be an isomorphism

Conclusion: $\pi_0^{\mathbb{A}^1}$ is not birational inv.

Example of a scheme $\underline{X}$ s.t

$\mathrm{Sing}_*(X)$ is not $\mathbb{A}^1$-local.

X smooth projective variety over $\mathbb{C}$ s.t

(i) $S(X) \neq S^2(X)$

(ii) $S(X) \longrightarrow \pi_0^{\mathbb{A}^1}(X)$ is not a mono.

(iii) $\mathrm{Sing}_*(X)$ is not $\mathbb{A}^1$-local

Sect. 4.1 : Const 4.5 : (4)

(Anant thesis.

↳ ruled case.