

I. A^1 -connected components (A refinement of Asok - Morel's theorem)

[Balwe - Hogadi - Sawant]

Theorem 1: $\mathcal{F} \in \text{Shv}_{\text{Nis}}(\text{Sm}_k) \iff \Delta^{\text{op}} \text{Shv}_{\text{Nis}}(\text{Sm}_k)$

$S(\mathcal{F})$ sheaf of A^1 -chain connected components

Then:

• $L(\mathcal{F}) := \lim_n S^n(\mathcal{F})$ is A^1 invariant

• If Morel's conjecture is true ($\pi_0^{A^1}(X)$ is A^1 -invariant) for \mathcal{F} , then

$$\pi_0^{A^1}(\mathcal{F}) \xrightarrow{\sim} L(\mathcal{F})$$

Theorem 2: X proper scheme over k
 L finitely generated field extension of k

Then, for every positive integer n :

$$S(X) (\text{Spec}(L)) \cong S^n(X) (\text{Spec}(L))$$

As a consequence:

$$S(X) (\text{Spec}(L)) \xrightarrow{\sim} \pi_0^{A1}(X) (\text{Spec}(L))$$

AsoK - Morel 2.4.3 (X ^{proper} finite type + L/k separable)

Recall:

Sections of $S(F)$:

$$pr_1, pr_2: V \times_u V \longrightarrow V$$

$u \in \text{Sm}_k$,

$$S(F)(u) \simeq \left\{ s \in F(V) \mid \begin{array}{l} V \rightarrow u \text{ finite Nisnevich cover s.t.} \\ pr_i^*(s) \text{ are } A^1\text{-chain homotopic} \\ \text{after restriction to a Nisnevich} \\ \text{cover of } V \times_u V. \end{array} \right\}$$

$$\begin{array}{ccccc} & & & & \tilde{s} \\ & & & & \longrightarrow \\ & & & u & \longrightarrow S(F) \\ & & & \uparrow & \downarrow \\ & & & V & \downarrow \\ & & & \uparrow & \\ & & & V & \xrightarrow{s} F \\ & & & \downarrow & \\ & & & V \times_u V & \\ \exists \varphi & \xrightarrow{\quad} & & & \\ W & \xrightarrow{\text{Nisn}} & & & \end{array}$$

$$\oplus h = (h_1, \dots, h_n) \in F(W \times A^1) \quad \text{s.t.}$$

$$\sigma_0^*(h_{i+1}) = \sigma_1^*(h_i) \quad 1 \leq i \leq n-1$$

$$\sigma_0^*(h_1) = \psi^* \underset{\parallel}{\text{pr}_1^*} s, \quad \sigma_1^*(h_n) = \psi^* \underset{\parallel}{\text{pr}_2^*} s.$$

$$\sigma_0, \sigma_1: W \rightarrow W \times A^1$$

$$w \mapsto (w, 0)$$

$$\text{pr}_1^* s \big|_W$$

$$\text{pr}_2^* s \big|_W$$

Remark:

- If t_1, t_2 are A^1 -chain homotopic, they map to the same element of $S(F)(u)$.
- If $t_1, t_2 \in F(u)$ s.t. they map to the same element in $S(F)(u)$ then $\exists v \xrightarrow{\text{NIS}} u$ s.t. $t_1|_v$ and $t_2|_v$ are A^1 -chain homotopic.

The description of sections of $S(F)$ applied to Al'_u gives rise to Al^1 -ghost homotopies:

Defⁿ: $F \in \text{Shv}_{\text{Nis}}(\text{Sm}_k)$, $U \in \text{Sm}_k$. An Al^1 -ghost homotopy consist of:

$$H := \left(\begin{array}{c} V \longrightarrow Al'_u \\ \text{Nis} \\ \text{cover} \end{array}, \begin{array}{c} W \longrightarrow V \times_{Al'_u} V \\ \text{Nis} \\ \text{cover} \end{array}, h, h^W \right)$$

- $h \in F(V)$ ($\cong h: V \rightarrow F$)

- h^W is an Al^1 -chain homotopy connecting:

$$W \longrightarrow V \times_{A^1_u} V \begin{array}{c} \xrightarrow{pr_1} \\ \xrightarrow{pr_2} \end{array} V \xrightarrow{h} \mathcal{F}$$

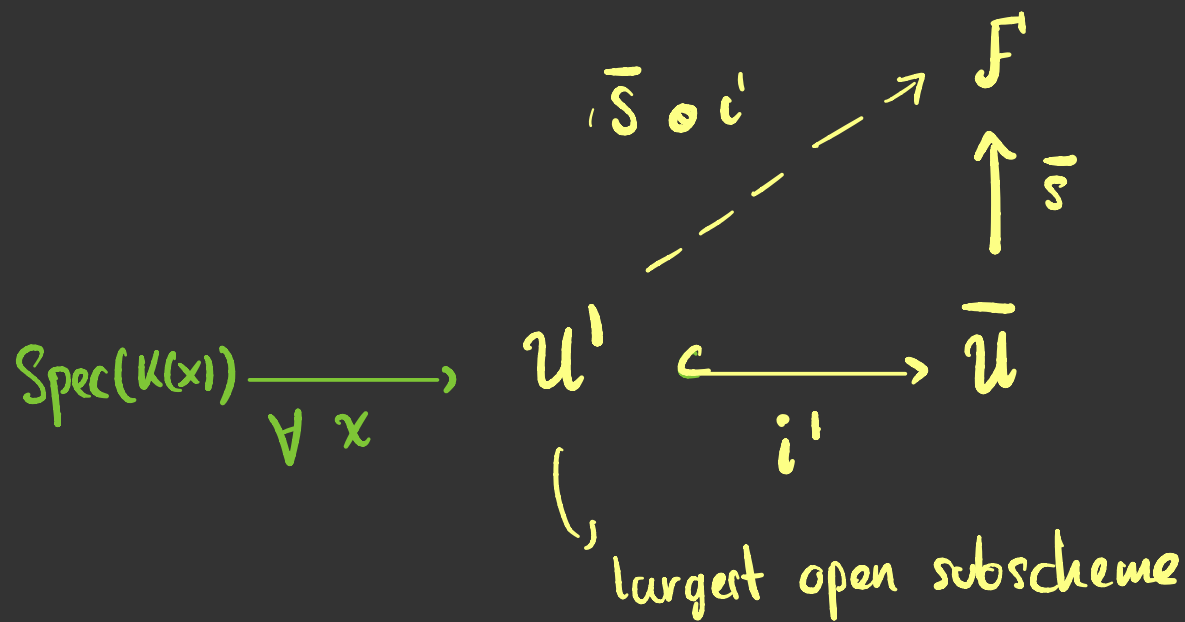
Lemma^{*}: Let \mathcal{F} sheaf over Sm_k , then $S(\mathcal{F}) = S^2(\mathcal{F})$

\Leftrightarrow $\forall u$ smooth Henselian local, if $t_1, t_2 \in \mathcal{F}(u)$ are ghost-homotopic then they are A^1 -chain homotopic.

Defⁿ \mathcal{F} sheaf over $\mathbb{A}^1_{\mathbb{K}}$, \mathcal{F} is said almost proper if

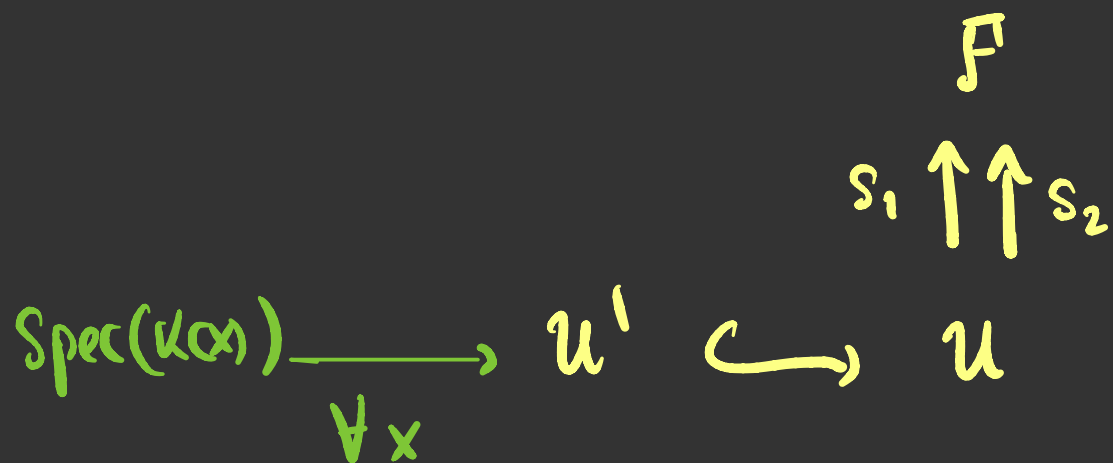
(AP1) $\forall U$ smooth irred variety of $\dim \leq 2$ and
 $s: U \rightarrow \mathbb{A}^1_{\mathbb{K}}$, then $\exists \bar{U}$ smooth projective,
 $i: U \dashrightarrow \bar{U}$ birrational and $\bar{s}: \bar{U} \rightarrow \mathbb{A}^1_{\mathbb{K}}$

that "extends s on points":



$$\bar{s} \circ i^{-1} \circ x = s|_{U^{-1}(x)}$$

(AP2) U smooth, irreducible curve over k , U' open subscheme of U and $s_1, s_2 : U \rightarrow F$ s.t. $s_1|_{U'} = s_2|_{U'}$ then $s_1 \sim_0 s_2$.



Lemma: \mathcal{F} almost proper sheaf on $S_m \kappa$

\mathcal{U} smooth curve over κ .

$x: \text{Spec}(\kappa) \rightarrow \mathcal{U}$ κ -rational point.

$\mathcal{U}' = \mathcal{U} - \{x\}$ open subscheme of \mathcal{U}

If $f, g: \mathcal{U} \rightarrow \mathcal{F}$ s.t. $f|_{\mathcal{U}'}$ and $g|_{\mathcal{U}'}$ are A^1 -chain homotopic

then $f \circ x, g \circ x$ are A^1 -chain homotopic.

Proof: "François talk", Lemma 3.7 [BHS]

Lemma: F almost proper $\Rightarrow \underline{S(F)}$ is also almost proper.

Proof: $\left[\begin{array}{l} \text{(AP2) } U \text{ smooth, irreducible curve over } k, U' \text{ open} \\ \text{subscheme of } U \text{ and } s_1, s_2 : U \rightarrow F \\ \text{s.t. } s_1|_{U'} = s_2|_{U'} \text{ then } s_1 \sim_0 s_2. \end{array} \right]$

Let U smooth irred curve over k , $U' \subseteq U$ open subscheme
and $s_1, s_2 : U \rightarrow S(F)$ s.t. $s_1|_{U'} = s_2|_{U'}$.

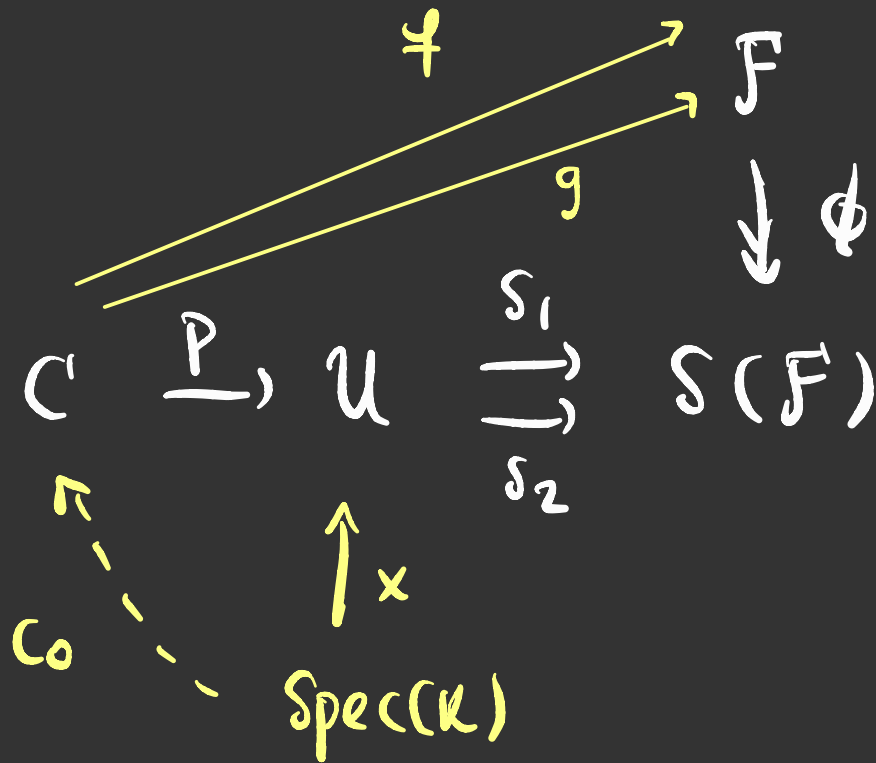
w.l.o.g we may assume $U - U'$ consist of a
single closed point x and suppose also x is
rational (modulo base change). $x : \text{Spec}(k) \rightarrow U$.

We need to prove that $s_1 \circ x = s_2 \circ x$.

We know $\phi : F \rightarrow S(F)$ is epimorphism. So

$\exists C \xrightarrow{P} U$ Nisnevich covering s.t. $s_1 \circ \rho, s_2 \circ \rho$
 lift to F , $\exists f, g : C \rightarrow F$ s.t

$$\begin{aligned} \phi \circ f &= s_1 \circ \rho \\ \phi \circ g &= s_2 \circ \rho \end{aligned}$$



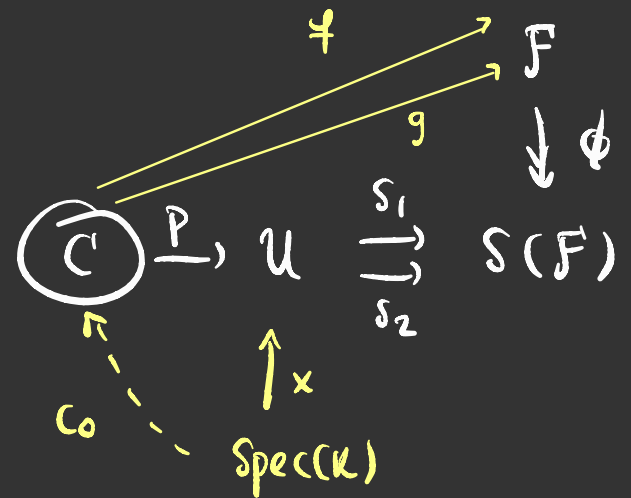
p is completely decomposed at x st $x: \text{Spec}(K) \rightarrow U$
 lifts to C via $c_0: \text{Spec}(K) \rightarrow C$.

$$p \circ c_0 = x.$$

rr

If we manage to prove that $f_0 \circ c_0$ and $g_0 \circ c_0$
 are A_1' -chain htpic, we're done!

$$\begin{aligned} s_1 \circ x &= s_1 \circ p \circ c_0 = \phi \circ f_0 \circ c_0 \\ &= \phi \circ g_0 \circ c_0 \\ &= s_2 \circ p \circ c_0 \\ &= s_2 \circ x \quad \underline{\underline{=}} \end{aligned}$$



C can be taken to be a curve (Why?) (OK! by dim reasons and p étale u ^{irred} curve).

You know $s_1|_{u'} = s_2|_{u'}$

$$\begin{array}{ccccc}
 V & \xrightarrow{q} & C & \xrightarrow{f} & F \\
 \downarrow & & \downarrow p & \xrightarrow{g} & \downarrow \phi \\
 u' & \hookrightarrow & u & \xrightarrow{s_1} & S(F) \\
 & & & \xrightarrow{s_2} &
 \end{array}$$

$\exists q: V \rightarrow C$ nisnevich such that

$f \circ q$ and $g \circ q$ is A^1 -chain

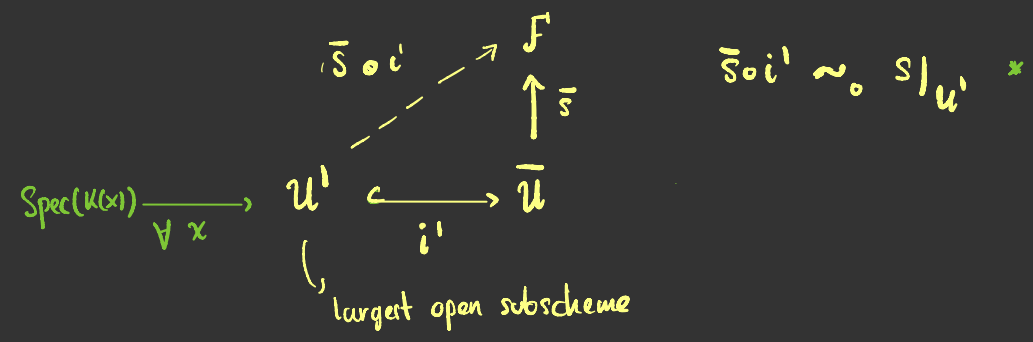
homotopic

Consider K function field of C together
with $\eta: \text{Spec}(K) \rightarrow F$. Since $V \xrightarrow{g} C$
is Nisnevich you can lift η to V so that
 $f \circ \eta$ and $g \circ \eta$ are A^1 -chain htpic.

$\exists C' \hookrightarrow C$ s.t. $f|_{C'}$ and $g|_{C'}$ are
 A^1 -chain htpic. (Lemma 3.7), by Lemma you
conclude that $f \circ \zeta_0$ and $g \circ \zeta_0$ are A^1 -chain
htpic.

(AP1) $\forall U$ smooth irred variety of $\dim \leq 2$ and
 $s: U \rightarrow F$, then $\exists \bar{U}$ smooth projective,
 $i: U \dashrightarrow \bar{U}$ birrational and $\bar{s}: \bar{U} \rightarrow F$

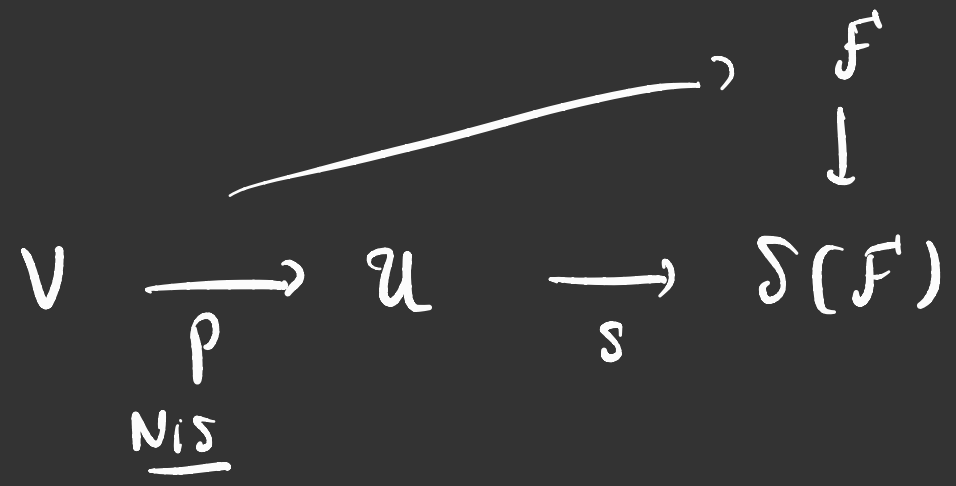
that "extends s on points":



Suppose U smooth, irred variety $\dim \leq 2$ and

$$s: U \rightarrow S(F)$$

Since $\phi: F \rightarrow S(F)$ is an epi, $\exists p: V \xrightarrow{Nis} U$ s.t



s.o. p lift to F . $\exists U' \subseteq U$ s.t. $s|_{U'}$ also

lift to \mathcal{F} , say $t: u' \rightarrow \mathcal{F}$. So

$\exists \bar{u}$ smooth projective, $i: u' \dashrightarrow \bar{u}$ birational

$$\downarrow \quad \bar{t}: \bar{u} \rightarrow \mathcal{F}$$

(that "extends t on points").

Let u'' largest open subscheme of u' s.t. i is
represented by $i': u'' \rightarrow \bar{u}$ (Recall

$$\bar{t} \circ i' \sim_0 t|_{u''})$$

Claim: $\phi \circ \bar{t}$ is the section you're looking for.

We have $i' : U \dashrightarrow \bar{U}$ rational, let U''' largest open subscheme of U s.t. $i' : U \dashrightarrow \bar{U}$ is rep

by $i'' : U''' \rightarrow \bar{U}$ you have $U'' \subseteq U'''$

Want to show $\forall x \in U'''$: $\left(\begin{array}{c} \bar{s} \\ \{ \\ \bar{t} \end{array} \right)$

$$\phi \circ \bar{t} \circ i'' \circ x = s|_{U'''} \circ x$$

If $x \in U'' \subseteq U'''$, you know:

$$\bar{t} \circ i' \circ x = t|_{U''} \circ x$$

$$\phi \circ \bar{t} \circ i' \circ x = \phi \circ t|_{U''} \circ x$$

$$= s|_{U''} \circ x = s \circ x$$

If $x \in U''' \setminus U''$. U'' is open dense in U

$\exists \mathcal{D} \hookrightarrow U'''$ locally closed immersion

\mathcal{D} contains x and intersects U''

z generic point of \mathbb{D} , $J(z) \in u''$ and
use "last case".



Theorem \mathcal{F} almost proper sheaf, then for any field extension K of k

$$S(\mathcal{F})(\text{Spec } K) \cong S^n(\mathcal{F})(\text{Spec } K) \quad \forall n \geq 1.$$

Proof: Assume $K = k$ (base change) by last lemma

it suffices to show

$$S(\mathcal{F})(\text{Spec}(k)) = S^2(\mathcal{F})(\text{Spec}(k))$$

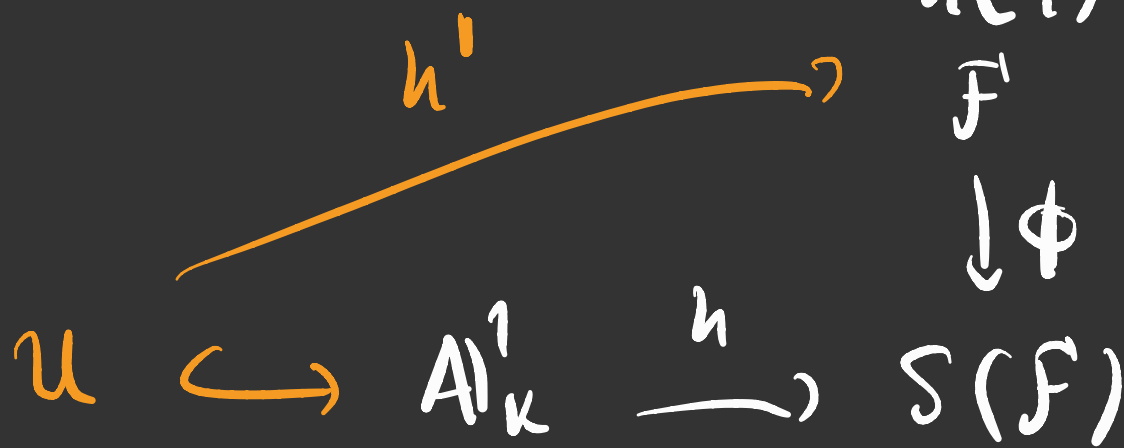
$$S(\mathcal{F}) \longrightarrow S^2(\mathcal{F})$$

If Idea take x, y s.t map to same thing in S^2
 $\Rightarrow x = y \quad \neq$

Let $x, y \in S(F)(\text{Spec}(K))$ and suppose

$h: A'_K \rightarrow S(F)$ s.t $h(0) = x$

$h(1) = y$



$\exists U$ open in A'_K and $h': U \rightarrow F$

such that $h|_U = \phi \circ h'$.

Since F is almost proper by (AP1) (relative to h') (find proj, bir, a section over proj)

$\exists \bar{h} : \mathbb{P}_k^1 \longrightarrow F$ extending h' on points

means if

$$\begin{array}{ccccc} U & \hookrightarrow & \mathbb{A}_k^1 & \hookrightarrow & \mathbb{P}_k^1 \\ & & & \searrow & \uparrow \\ & & & & i \end{array}$$

then $\forall x \in U \quad \bar{h} \circ i \circ x = h' \circ x$

use this to the generic point of U to get U' open subscheme of U s.t

$$\bar{h}|_{U'} = \bar{h} \circ i|_{U'} = h'|_{U'}$$

take $\tilde{h} := \phi \circ \bar{h}|_{A^1_k} : A^1 \rightarrow \mathcal{S}(F)$

and $\tilde{h}|_{U'} = (\phi \circ \bar{h}|_{A^1_k})|_{U'} = \phi \circ \bar{h}|_{U'}$
 $= \phi \circ h'|_{U'}$
 $= h|_{U'}$

By (AP2) $\implies \tilde{h} \sim \circ h$

$$h(0) = x = (\phi \circ \tilde{h})(0:1)$$

$$h(1) = y = (\phi \circ \tilde{h})(1:1)$$

x and y are images under ϕ of A^1 -htpic
morphism $\text{Spec}(k) \rightarrow \mathbb{P}^1$ by def of S

$$x = y.$$

□

Corollary: \mathcal{F} almost proper sheaf then

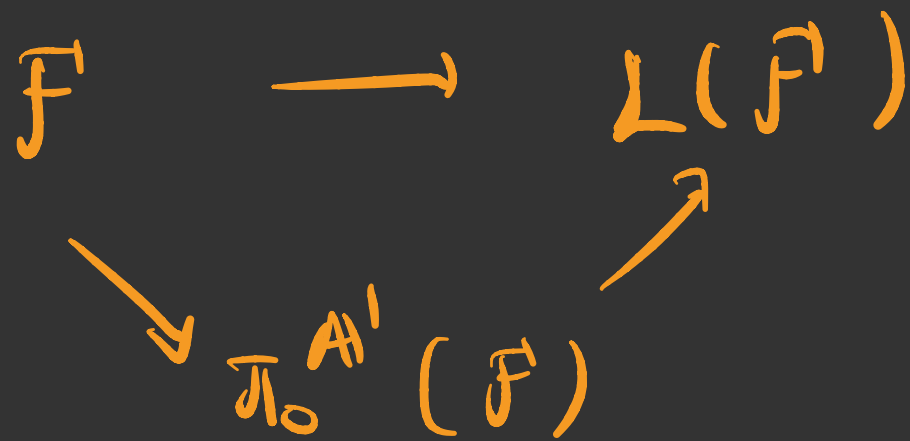
$$S(\mathcal{F})(\text{Spec } \mathcal{K}) \xrightarrow{\sim} \pi_0^{\text{Al}^1}(\text{Spec } \mathcal{K}) \quad \text{for}$$

any $\mathcal{K} | \kappa$ finitely generated. In particular
this holds for X proper, finite type over κ .

Recall

$$S(\mathcal{F})(\mathcal{K}) \xrightarrow{\sim} L(\mathcal{F})(\mathcal{K})$$

$$\begin{array}{ccc} & \uparrow & \nearrow \\ \phi & \mathcal{F} & \end{array}$$



uniquely
factors

II. Morel's conjectures for non-uniruled surfaces

Conjecture 1: (Morel) \mathcal{X} simplicial sheaf, then $\pi_0^{A^1}(\mathcal{X})$ is A^1 -invariant.

Conjecture 2: X smooth scheme over k , the natural epimorphism $\mathcal{P}(X) \longrightarrow \pi_0^{A^1}(X)$ is an iso.

Goal: Conjectures 1 and 2 hold for non-uniruled surfaces over k .



Lemma*: Let \mathcal{F} sheaf over Sm_k , then $S(\mathcal{F}) = S^2(\mathcal{F})$

$\Leftrightarrow \forall U$ smooth Henselian local, if $t_1, t_2 \in \mathcal{F}(U)$ are ghost-homotopic then they are \mathbb{A}^1 -chain homotopic.

\rightsquigarrow hold ($\dim U > 0$)

Idea:

+ reduction to 1-dimensional schemes.

Def: X scheme over k , $f: Y \rightarrow X$ is a \mathbb{P}^1 -fibration if f is smooth, proper and $\forall x \in X$, $f^{-1}(x)$ is a rational curve.

Lemma: E, B varieties over k

$\pi: E \rightarrow B$ smooth, projective over k

$\forall b \in B$, $\pi^{-1}(b) = E_b$ isomorphic to \mathbb{P}_b^1

Then π is an étale locally trivial fiber bundle.

Proposition X reduced, separated, proper and 1-dim
over k , Then

$$S(X) \cong S^2(X).$$

↓
Thm

X proper, non uniruled surface over
 k then $S(X) = S^2(X)$