I. $\mathbb{A}^1$ - connected components (A refinement of Asok - Morel's theorem)

[Balwe - Hogadi - Sawant]

Theorem 1: $\mathcal{F} \in Shv_{Nis}(Sm_k) \hookrightarrow \Delta^{op} Shv_{Nis}(Sm_k)$

$S(\mathcal{F})$ sheaf of $\mathbb{A}^1$- chain connected components

Then:

- $\mathcal{L}(\mathcal{F}) := \lim_n S^n(\mathcal{F})$ is $\mathbb{A}^1$ invariant

- If Morel's conjecture is true (Conj: $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant) for $\mathcal{F}$, then

$$\pi_0^{\mathbb{A}^1}(\mathcal{F}) \xrightarrow{\sim} \mathcal{L}(\mathcal{F})$$

Theorem 2: X proper scheme over k

L finitely generated field extension of k

Then, for every positive integer n:

$$S(X)(\mathrm{Spec}(L)) \cong S^n(X)(\mathrm{Spec}(L))$$

As a consequence:

$$S(X)(\mathrm{Spec}(L)) \xrightarrow{\sim} \pi_0^{\mathbb{A}^1}(X)(\mathrm{Spec}(L))$$

Asok - Morel 2.4.3 (X ~~finite type~~ proper + L/k separable)

Recall:

<u>Sections of $S(\mathcal{F})$:</u>

$pr_1, pr_2 : V \times_{\mathcal{U}} V \longrightarrow V$

$\mathcal{U} \in Sm_k$,

$$S(\mathcal{F})(\mathcal{U}) \simeq \left\{ s \in \mathcal{F}(V) \;\middle/\; \begin{array}{l} V \to \mathcal{U} \text{ finite Nisnevich cover s.t} \\ pr_i^*(s) \text{ are } \mathbb{A}^1\text{- chain homotopic} \\ \text{after restriction to a Nisnevich} \\ \text{cover of } V \times_{\mathcal{U}} V. \end{array} \right\}$$

$$\begin{tikzcd}
 & & \mathcal{U} \arrow[r, "\tilde{s}"] & S(\mathcal{F}) \arrow[d, twoheadrightarrow] \\
 & V \arrow[ur] & & \\
W \arrow[r, "\exists \varphi", "\mathrm{Nisn}"'] & V \times_{\mathcal{U}} V \arrow[u, "pr_2"] \arrow[r, "pr_1"'] & V \arrow[uu] \arrow[r, "s"] & \mathcal{F}
\end{tikzcd}$$

⊕ $h = (h_1, \dots, h_n) \in \mathcal{F}(W \times \mathbb{A}^1)$ s.t

$$\sigma_0^*(h_{i+1}) = \sigma_1^*(h_i) \qquad 1 \le i \le n-1$$

$$\sigma_0^*(h_1) = \underset{\underset{pr_1^* s|_W}{\shortparallel}}{\varphi^* pr_1^* s}, \qquad \sigma_1^*(h_n) = \underset{\underset{pr_2^* s|_W}{\shortparallel}}{\varphi^* pr_2^* s}.$$

$\sigma_0, \sigma_1 : W \longrightarrow W \times \mathbb{A}^1$

$w \mapsto (w, \overset{0}{1})$

Remark:

- If t_1, t_2 are $\mathbb{A}^1$-chain homotopic, they map to the same element of $S(F)(U)$.

- If $t_1, t_2 \in \mathcal{F}(U)$ s.t they map to the same element in $S(F)(U)$ then $\exists\, V \xrightarrow{Nis} U$ s.t $t_1|_V$ and $t_2|_V$ are $\mathbb{A}^1$-chain homotopic.

The description of sections of $S(\mathcal{F})$ applied to $\mathbb{A}^1_U$ gives rise to $\mathbb{A}^1$- ghost homotopies:

<u>**Def^n**</u>: $\mathcal{F} \in Shv_{Nis}(Sm_k)$, $U \in Sm_k$. An <u>$\mathbb{A}^1$-ghost homotopy</u> consist of:

$$\mathcal{H} := \left(V \xrightarrow[\text{Nisn cover}]{} \mathbb{A}^1_U , \; W \xrightarrow[\text{Nisn cover}]{} V \times_{\mathbb{A}^1_U} V , \; h, \; h^W \right)$$

- $h \in \mathcal{F}(V) \quad (\cong h : V \longrightarrow \mathcal{F})$
- h^W is an $\mathbb{A}^1$- chain homotopy connecting:

$$W \longrightarrow V \times_{\mathbb{A}^1_{\mathcal{U}}} V \overset{pr_1}{\underset{pr_2}{\rightrightarrows}} V \xrightarrow{h} \mathcal{F}$$

Lemma*: Let $\mathcal{F}$ sheaf over Sm_k, then $S(\mathcal{F}) = S^2(\mathcal{F})$

$\iff$ $\forall\, \mathcal{U}$ smooth Henselian local, if $t_1, t_2 \in \mathcal{F}(\mathcal{U})$ are ghost-homotopic then they are $\mathbb{A}^1$-chain homotopic.

Def^n $\mathcal{F}$ sheaf over Sm_k, $\mathcal{F}$ is said <u>almost proper</u> if

(AP1) $\forall\, U$ smooth irred variety of dim ≤ 2 and $s : U \to \mathcal{F}$, then $\exists\, \overline{U}$ smooth projective, $i : U \dashrightarrow \overline{U}$ birational and $\bar{s} : \overline{U} \longrightarrow \mathcal{F}$ that "extends s on points":

$$\begin{tikzcd}
 & & \mathcal{F} \\
\mathrm{Spec}(k(x)) \arrow[r, "\forall x"'] & U' \arrow[ur, dashed, "\bar{s}\circ i'"] \arrow[r, hook, "i'"'] & \overline{U} \arrow[u, "\bar{s}"']
\end{tikzcd}$$

U' : largest open subscheme

$\bar{s}\circ i' \sim_0 s|_{U'}$ *

$\bar{s}\circ i' \circ x = s|_{U'} \circ x$

(AP2) $\mathcal{U}$ smooth, irreducible curve over k, $\mathcal{U}'$ open subscheme of $\mathcal{U}$ and $s_1, s_2 : \mathcal{U} \to F$ s.t $s_1|_{\mathcal{U}'} = s_2|_{\mathcal{U}'}$ then $s_1 \sim_0 s_2$.

$$\begin{tikzcd}
 & & F \\
\mathrm{Spec}(k(x)) \arrow[r, "\forall x"'] & \mathcal{U}' \arrow[r, hook] & \mathcal{U} \arrow[u, shift left, "s_1"] \arrow[u, shift right, "s_2"']
\end{tikzcd}$$

Lemma: $\mathcal{F}$ almost proper sheaf on Sm_k

$\mathcal{U}$ smooth curve over k.

$x : \operatorname{Spec}(k) \longrightarrow \mathcal{U}$ k-rational point.

$\mathcal{U}' = \mathcal{U} \setminus \{x\}$ open subscheme of $\mathcal{U}$

If $f, g : \mathcal{U} \to \mathcal{F}$ s.t $f|_{\mathcal{U}'}$ and $g|_{\mathcal{U}'}$ are $\mathbb{A}^1$-chain homotopic

then $f \circ x$, $g \circ x$ are $\mathbb{A}^1$-chain homotopic.

Proof: "François talk", Lemma 3.7 [BHS]

Lemma: F almost proper $\Rightarrow$ $S(F)$ is also almost proper.

Proof: [(AP2) $\mathcal{U}$ smooth, irreducible curve over k, $\mathcal{U}'$ open subscheme of $\mathcal{U}$ and $s_1, s_2 : \mathcal{U} \to F$ s.t $s_1|_{\mathcal{U}'} = s_2|_{\mathcal{U}'}$ then $s_1 \sim_0 s_2$.]

Let $\mathcal{U}$ smooth irred curve over k, $\mathcal{U}' \subseteq \mathcal{U}$ open subscheme and $s_1, s_2 : \mathcal{U} \to S(F)$ s.t $s_1|_{\mathcal{U}'} = s_2|_{\mathcal{U}'}$.

W.l.o.g we may assume $\mathcal{U} \setminus \mathcal{U}'$ consist of a single closed point x and suppose also x is rational (modulo base change). $x : \mathrm{Spec}(k) \to \mathcal{U}$.

We need to prove that $s_1 \circ x = s_2 \circ x$.

We know $\phi : F \longrightarrow S(F)$ is epimorphism. So

$\exists\ C \xrightarrow{p} U$ Nisnevich covering s.t $s_1 \circ p$, $s_2 \circ p$

lift to F, $\exists\ f, g : C \longrightarrow F$ s.t

$$\phi \circ f = s_1 \circ p$$

$$\phi \circ g = s_2 \circ p$$

$$\begin{tikzcd}
 & & F \arrow[d, "\phi"] \\
C \arrow[urr, shift left, "f"] \arrow[urr, shift right, "g"'] \arrow[r, "p"] & U \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & S(F) \\
 & \mathrm{Spec}(k) \arrow[u, "x"'] \arrow[ul, dashed, bend left, "c_0"] &
\end{tikzcd}$$

p is completely decomposed at x st $x : \mathrm{Spec}(k) \longrightarrow U$ lifts to C via $c_0 : \mathrm{Spec}(k) \longrightarrow C$.

$$p \circ c_0 = x.$$

If we manage to prove that $f \circ c_0$ and $g \circ c_0$ are $\mathbb{A}^1$-chain htpic, we're done!

$$\begin{tikzcd}
 & & & F \arrow[d, "\phi"] \\
C \arrow[urrr, shift left, "f"] \arrow[urrr, shift right, "g"'] \arrow[r, "p"] & U \arrow[rr, shift left, "s_1"] \arrow[rr, shift right, "s_2"'] & & S(F) \\
 & \mathrm{Spec}(k) \arrow[u, "x"'] \arrow[ul, dashed, "c_0"] & &
\end{tikzcd}$$

$$\begin{aligned}
s_1 \circ x = s_1 \circ p \circ c_0 &= \phi \circ f \circ c_0 \\
&= \phi \circ g \circ c_0 \\
&= s_2 \circ p \circ c_0 \\
&= s_2 \circ x
\end{aligned}$$

C can be taken to be a curve (Why?) (ok! by dim reasons and p étale, $\mathcal{U}$ irred curve).

You know $s_1|_{\mathcal{U}'} = s_2|_{\mathcal{U}'}$

$$\begin{tikzcd}
V \arrow[r, "q"] \arrow[d] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "p"] & F \arrow[d, "\phi"] \\
\mathcal{U}' \arrow[r, hook] & \mathcal{U} \arrow[r, shift left, "s_1"] \arrow[r, shift right, "s_2"'] & S(F)
\end{tikzcd}$$

pullback!

$\exists$ $q : V \to C$ Nisnevich such that $f \circ q$ and $g \circ q$ is $\mathbb{A}^1$-chain homotopic

Consider K function field of C together with $\eta: \mathrm{Spec}(K) \longrightarrow F$. Since $V \xrightarrow{q} C$ is Nisnevich you can lift η to V so that $f \circ \eta$ and $g \circ \eta$ are $\mathbb{A}^1$-chain htpic.

$\exists\, C' \hookrightarrow C$ s.t $f|_{C'}$ and $g|_{C'}$ are $\mathbb{A}^1$-chain htpic. (Lemma 3.7), by Lemma you conclude that $f \circ c_0$ and $g \circ c_0$ are $\mathbb{A}^1$-chain htpic.

(AP1) $\forall$ $\mathcal{U}$ smooth irred variety of dim ≤ 2 and $s : \mathcal{U} \to \mathcal{F}$, then $\exists$ $\bar{\mathcal{U}}$ smooth projective, $i : \mathcal{U} \dashrightarrow \bar{\mathcal{U}}$ birational and $\bar{s} : \bar{\mathcal{U}} \to \mathcal{F}$ that "extends s on points":

$$
\begin{tikzcd}
 & & \mathcal{F} \\
\mathrm{Spec}(\kappa(x)) \arrow[r, "\forall x"'] & \mathcal{U}' \arrow[ur, dashed, "\bar{s} \circ i'"] \arrow[r, hook, "i'"'] & \bar{\mathcal{U}} \arrow[u, "\bar{s}"']
\end{tikzcd}
$$

$\mathcal{U}'$: largest open subscheme

$\bar{s} \circ i' \sim_0 s|_{\mathcal{U}'}$ *

Suppose $\mathcal{U}$ smooth, irred variety dim ≤ 2 and

$$s : \mathcal{U} \to S(\mathcal{F})$$

Since $\phi : \mathcal{F} \to S(\mathcal{F})$ is an epi, $\exists$ $p : V \xrightarrow{Nis} \mathcal{U}$ s.t

$$
\begin{tikzcd}
 & & \mathcal{F} \arrow[d] \\
V \arrow[urr] \arrow[r, "p"', "Nis"'] & \mathcal{U} \arrow[r, "s"'] & S(\mathcal{F})
\end{tikzcd}
$$

so p lift to $\mathcal{F}$. $\exists$ $\mathcal{U}' \subseteq \mathcal{U}$ s.t $s|_{\mathcal{U}'}$ also lift to $\mathcal{F}$, say $t : \mathcal{U}' \longrightarrow \mathcal{F}$. So

$\exists\ \overline{\mathcal{U}}$ smooth projective, $i : \mathcal{U}' \dashrightarrow \overline{\mathcal{U}}$ birational

$\bar{t} : \overline{\mathcal{U}} \longrightarrow \mathcal{F}$

(that "extends t on points").

Let $\mathcal{U}''$ largest open subscheme of $\mathcal{U}'$ s.t i is represented by $i' : \mathcal{U}'' \longrightarrow \overline{\mathcal{U}}$ (Recall

$\bar{t} \circ i' \sim_0 t|_{\mathcal{U}''}$)

Claim : $\phi \circ \bar{t}$ is the section you're looking for.

We have $i' : \mathcal{U} \dashrightarrow \bar{\mathcal{U}}$ rational, let $\mathcal{U}'''$ largest open subschem of $\mathcal{U}$ s.t $i' : \mathcal{U} \dashrightarrow \bar{\mathcal{U}}$ is rep by

$$i'' : \mathcal{U}''' \longrightarrow \bar{\mathcal{U}}$$

you have $\mathcal{U}'' \subseteq \mathcal{U}'''$

Want to show $\forall x \in \mathcal{U}'''$:

$$\left(\bar{s} \overset{\wr}{=} \bar{t} \right)$$

$$\phi \circ \bar{t} \circ i'' \circ x = s|_{\mathcal{U}'''} \circ x$$

If $x \in U'' \subseteq U'''$, you know:

$$\bar{t} \circ i' \circ x = t|_{U''} \circ x$$

$$\phi \circ \bar{t} \circ i' \circ x = \phi \circ t|_{U''} \circ x$$
$$= s|_{U''} \circ x = s \circ x$$

If $x \in U''' \setminus U''$.

U'' is open dense in U

$\exists$ $J : D \hookrightarrow U'''$ locally closed immersion

$J(D)$ contains x and intersects U''

z generic point of D, $J(z) \in \mathcal{U}$" and use "last case".

$\square$

Theorem $\mathcal{F}$ almost proper sheaf, then for any field extension K of κ

$$S(\mathcal{F})(\operatorname{Spec} K) \simeq S^n(\mathcal{F})(\operatorname{Spec} K) \quad \forall n \geq 1.$$

Proof: Assume $K = \kappa$ (base change) by last lemma it suffices to show

$$S(\mathcal{F})(\operatorname{spec}(\kappa)) = S^2(\mathcal{F})(\operatorname{spec}(\kappa))$$

$$S(\mathcal{F}) \twoheadrightarrow S^2(\mathcal{F})$$

"Idea" take x, y s.t map to same thing in S^2

$\Rightarrow x = y$ $\dashv$

Let $x, y \in S(\mathcal{F}')(\mathrm{Spec}(k))$ and suppose

$h : \mathbb{A}^1_k \longrightarrow S(\mathcal{F})$ s.t $h(0) = x$

$h(1) = y$

$$\begin{tikzcd}
 & & \mathcal{F}' \arrow[d, "\phi"] \\
U \arrow[r, hook] \arrow[urr, bend left, "h'"] & \mathbb{A}^1_k \arrow[r, "h"] & S(\mathcal{F})
\end{tikzcd}$$

$\exists$ U open in $\mathbb{A}^1_k$ and $h' : U \longrightarrow \mathcal{F}'$

such that $h|_U = \phi \circ h'$.

Since $\mathcal{F}$ is almost proper by (AP1) (relative to h') (find proj, bir, a section over proj)

$\exists\ \bar{h} : \mathbb{P}^1_k \longrightarrow \mathcal{F}$ extending h' on points

means if

$$\begin{tikzcd} U \arrow[r, hook] \arrow[rr, bend right, "i"'] & \mathbb{A}^1_k \arrow[r, hook] & \mathbb{P}^1_k \end{tikzcd}$$

then $\forall\, x \in U$ $\quad \bar{h} \circ i \circ x = h' \circ x$

use this to the generic point of $\mathcal{U}$ to get $\mathcal{U}'$ open subscheme of $\mathcal{U}$ s.t

$$\bar{h}|_{\mathcal{U}'} = \bar{h} \circ i|_{\mathcal{U}'} = h'|_{\mathcal{U}'}$$

take $\tilde{h} := \phi \circ \bar{h}|_{\mathbb{A}^1_K} : \mathbb{A}^1 \longrightarrow S(F)$

and $\tilde{h}|_{\mathcal{U}'} = \left(\phi \circ \bar{h}|_{\mathbb{A}^1_K}\right)|_{\mathcal{U}'} = \phi \circ \bar{h}|_{\mathcal{U}'}$

$$= \phi \circ h'|_{\mathcal{U}'}$$

$$= h|_{\mathcal{U}'}$$

By (AP2) $\Longrightarrow$ $\tilde{h} \sim_{\circ} h$

$$h(0) = x = (\phi \circ \bar{h})(0:1)$$
$$h(1) = y = (\phi \circ \bar{h})(1:1)$$

x and y are images under ϕ of $\mathbb{A}^1$-htpic morphism $\operatorname{Spec}(k) \longrightarrow F$ by def of S

$x = y$.

$\boxtimes$

Corollary: $\mathcal{F}$ almost proper sheaf then

$$S(\mathcal{F})(\operatorname{Spec} K) \xrightarrow{\sim} \pi_0^{\mathbb{A}^1}(\operatorname{Spec} K) \quad \text{for}$$

any $K|_k$ finitely generated. In particular this holds for X proper, finite type over k.

Recall

$$\begin{tikzcd}
S(\mathcal{F})(K) \arrow[rr, "\sim"] & & L(\mathcal{F})(K) \\
\mathcal{F} \arrow[u, "\phi"] \arrow[urr] & &
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{F} \arrow[rr] \arrow[dr] & & L(\mathcal{F}) \\
& \pi_0^{\mathbb{A}^1}(\mathcal{F}) \arrow[ur] &
\end{tikzcd}
$$

uniquely
factors

II. Morel's conjectures for non-uniruled surfaces

Conjecture 1: (Morel) $\mathcal{X}$ simplicial sheaf, then $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant.

Conjecture 2: X smooth scheme over k, the natural epimorphism

$$S(X) \longrightarrow \pi_0^{\mathbb{A}^1}(X) \text{ is an iso.}$$

Goal: Conjectures 1 and 2 hold for non-uniruled surfaces over k.

Idea: **Lemma***: Let $\mathcal{F}$ sheaf over Sm_k, then $S(\mathcal{F}) = S^2(\mathcal{F})$ $\rightsquigarrow$ hold $(\dim \mathcal{U} > 0)$

$\Leftrightarrow$ $\forall\ \mathcal{U}$ smooth Henselian local, if $t_1, t_2 \in \mathcal{F}(\mathcal{U})$ are ghost-homotopic then they are $\mathbb{A}^1$-chain homotopic.

+ reduction to 1-dimensional schemes.

Def: X scheme over k, $f: Y \longrightarrow X$ is a **$\mathbb{P}^1$-fibration** if f is smooth, proper and $\forall x \in X$, $f^{-1}(x)$ is a rational curve.

Lemma: E, B varieties over k

$\pi: E \longrightarrow B$ smooth, projective over k

$\forall b \in B$, $\pi^{-1}(b) = E_b$ isomorphic to $\mathbb{P}^1_b$

Then π is an étale locally trivial fiber bundle.

Proposition X reduced, separated, proper and 1-dim over k, Then

$$S(X) \simeq S^2(X).$$

Thm X proper, non uniruled surface over k then $S(X) = S^2(X)$