$\mathbb{A}^1$-homotopies on ruled surfaces

(after Balwe - Sawant)

k: base field, alg. closed and of char. 0
$\mathcal{H}(k)$: unstable homotopy category
$\mathcal{X}$: simplicial Nisnevich sheaf of sets on Sm/k

Def $S(\mathcal{X}) := a_{Nis}(\pi_0 Sing_*^{\mathbb{A}^1}(\mathcal{X}))$

$\mathcal{Y}$: simplicial Nis. sheaf of sets on Sm/k
$\pi_0(\mathcal{Y})(U) := coeq(\mathcal{Y}_1(U) \rightrightarrows \mathcal{Y}_0(U))$
$Sing_*^{\mathbb{A}^1}(\mathcal{X})_n = \underline{Hom}(\Delta^n, \mathcal{X}_n)$
where $\Delta^n = Spec(k[x_0, \dots, x_n]/(\sum x_i = 1))$

Def $\pi_0^{\mathbb{A}^1}(\mathcal{X}) := a_{Nis} Hom_{\mathcal{H}(k)}(-, \mathcal{X})$ for $\mathcal{X} \in \mathcal{H}(k)$.

There is always a map $S(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$ which is not always an isomorphism.

- $S(\mathcal{X})(U) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(\mathcal{X})(U)$ if $U = Spec(F)$ for F/k fin. gen. and separable;
- there exists a threefold X such that $S(X)(U) \xrightarrow[\times]{\simeq} \pi_0^{\mathbb{A}^1}(X)(U)$ if $U = Spec(R)$ for R/k Henselian DVR

Thm If X/k smooth, projective, $\dim(X) \leq 2$, then $S(X)(U) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(X)(U)$ for each $U = Spec(R)$ where R/k Henselian local, $\dim(R) \leq 1$.

Easy to see for rational surfaces (Asok - Morel)
Proven for non-uniruled surfaces by Balwe, Hogadi, Sawant.

└ We are left with birationally uniruled surfaces over a curve C of genus > 0 $\Rightarrow$ C is $\mathbb{A}^1$-rigid.

Thm X/k smooth proj. surface, birationally ruled over a curve C of genus > 0, and X is not a minimal model, the natural epimorphism $S(X) \to S^2(X)$ is not an isomorphism, hence $Sing_*^{\mathbb{A}^1}(X)$ is not $\mathbb{A}^1$-local.

Compare with the following theorem.
Thm X/k smooth projective surface $\Rightarrow$ $\pi_0^{\mathbb{A}^1}(X) = S^2(X) = S^n(X)$, $\forall n \geq 2$.

To prove this theorems, one crucial ingredient is the following special case.

Thm If C/k smooth, proj. curve, of genus > 0, and $E \to C$ is a $\mathbb{P}^1$-bundle, and $X = Bl_p(E)$ for some $p \in |E|$, then $S(X)(U) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(X)(U)$ if $U = Spec(R)$ with R Henselian local, $\dim(R) \leq 1$, but $S(X) \not\simeq S^2(X)$.

Proof We will use a special case of the following lemma.

Lemma If $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ is a map of simplicial sheaves on Sm/k, s.t. $\mathcal{G}$ is $\mathbb{A}^1$-invariant, and $\mathcal{H}$ is a ghost homotopy of U in $\mathcal{F}$, then $\mathcal{H}$ respects the fibers of ϕ. Concretely if $\mathcal{H}: \mathbb{A}^1_U \to \mathcal{F}$ is a homotopy, this means

$$\begin{tikzcd}
\mathbb{A}^1_U \arrow[r, "\mathcal{H}"] \arrow[d] \arrow[dr, phantom, "//"] & \mathcal{F} \arrow[d, "\phi"] \\
U \arrow[r, dashed, "\exists \gamma"'] & \mathcal{G}
\end{tikzcd}$$

Fix $U = Spec(R)$ with R Henselian local, smooth.

√ This means that we can fix $\gamma: U \to C$, and consider sections $U \to X$ over γ.

Since a $\mathbb{P}^1$-bundle is étale-locally trivial, we have that $E_\gamma := E \times_{C,\gamma} U \simeq \mathbb{P}^1_U$

$$\begin{tikzcd}
 & X \arrow[d] \\
\{p\} \arrow[r, hook] & E \arrow[d] \\
U \arrow[r, "\gamma"'] \arrow[uur, dashed] & C
\end{tikzcd}$$

WLOG Assume that $E = \mathbb{P}^1_C = C \times \mathbb{P}^1$

Under this assumption, we can assume that

$$p = (c_0, (0:1))$$

for some point $c_0 \in C$.

WLOG Assume that γ maps the closed point of U to c_0, and that $\gamma(U) \neq \{c_0\}$.

└ Why? • If $c_0 \notin \gamma(U)$ then $X_\gamma := X \times_{C,\gamma} U \simeq \mathbb{P}^1_U$
$\Rightarrow$ Any two sections of $X_\gamma \to U$ will be $\mathbb{A}^1$-chain homotopic.
• If $\gamma(U) = \{c_0\}$ $\Rightarrow$ $X_\gamma = U \times T$ where T is the union of two copies of $\mathbb{P}^1$, intersecting transversely in a single point. Again, in this case any two sections of $X_\gamma \to U$ are $\mathbb{A}^1$-chain homotopic.

In this case, $X_\gamma = X(r_0)$ is the blow-up of $\mathbb{P}^1_U = E_\gamma = Proj(R[Y,Z])$ at the closed subscheme $V(I(r_0))$ where $I(r_0) = \langle Y, r_0 \rangle$, where $r_0 := \gamma^*(w)$ and w is a uniformizer of $\mathcal{O}_{C,c_0}$.

To study $S(X)(U)$, it suffices hence to understand homotopy classes of sections of $X(r_0) \to U$. Looking at the diagram

$$\begin{tikzcd}
X(r_0) \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{P}^1_U \arrow[r] \arrow[d] & E \simeq \mathbb{P}^1_C \arrow[d] \\
U \arrow[r, "\gamma"'] \arrow[u, dashed, bend right, "?"'] \arrow[uu, dashed, bend left=60, "\forall"] & C
\end{tikzcd}$$

we get an injective map

$$\left\{ \begin{matrix} \text{sections of} \\ X(r_0) \to U \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{sections of} \\ \mathbb{P}^1_U \to U \end{matrix} \right\}$$

whose image is given by those sections $\alpha: U \to \mathbb{P}^1_U$ s.t. $\alpha^*(I(r_0))$ is principal.

└ In this case, three possibilities can occur:

① $\alpha^*(I(r_0)) = \langle 1 \rangle \Rightarrow \alpha: U \to Spec(R[Z/Y]) \hookrightarrow \mathbb{P}^1_U$ is determined by $Z/Y \mapsto r$ for $r \in R$. In this case, we write $\alpha = \alpha_r$.

② $\alpha^*(I(r_0)) = \langle r_0 \rangle \Rightarrow \alpha: U \to Spec(R[Y/Z]) \hookrightarrow \mathbb{P}^1_U$, $r \leftarrow\!\shortmid Y/Z$ where $r_0 | r$. In this case, we set $\alpha = \beta_r$

③ $\alpha^*(I(r_0)) = \langle r \rangle$ for some $r \in R$ s.t. $r | r_0$ and $r_0 \nmid r \Rightarrow \alpha = \beta_{r'}$ with $\langle r \rangle = \langle r' \rangle$.

Then, the following happens:

- any two elements of the set $\{\alpha_r : r \in R\} \cup \{\beta_r : r_0 | r\}$ are $\mathbb{A}^1$-chain homotopic, by some homotopy which lifts to X_γ;
- if $r_1, r_2 \in R$ such that $r_1 | r_0$, $r_2 | r_0$ but $r_0 \nmid r_1$, $r_0 \nmid r_2$, then

$$\underbrace{\beta_{r_1} \underset{X_\gamma}{\sim} \beta_{r_2}} \iff \langle r_1 \rangle = \langle r_2 \rangle \text{ and } \underbrace{\frac{r_2}{r_1}} - 1 \in \sqrt{\langle r_1 \rangle} + \sqrt{\left\langle \frac{r_0}{r_1} \right\rangle} \subseteq \sqrt{\langle r_1, r_0/r_1 \rangle}$$

i.e. β_{r_1} and β_{r_2} are $\mathbb{A}^1$-chain homotopic by a homotopy lifting to X_γ.

Assuming this, we can conclude. Indeed:

- if R is a DVR, then
$$\sqrt{\langle r_1 \rangle} + \sqrt{\langle r_0/r_1 \rangle} = \sqrt{\langle r_1, r_0/r_1 \rangle}$$
if $r_1 | r_0$. Therefore any two sections of $X_\gamma \to U$ are chain homotopic, and we have that $S(X)(U) = S^2(X)(U) = \pi_0^{\mathbb{A}^1}(X)(U)$
- if $R = k\{x,y\}^h_{(x,y)}$, we can take
$$r_0 = x(x+y^2), \; r_1 = x, \; r_2 = x(y+1)$$
and we have that
$$\frac{r_2}{r_1} - 1 = y \notin \sqrt{\langle r_1 \rangle} + \sqrt{\left(\frac{r_0}{r_1}\right)} = \langle x, x+y^2 \rangle = \langle x, y^2 \rangle$$
$$\cap$$
$$\langle x, y \rangle = \sqrt{\left\langle r_1, \frac{r_0}{r_1} \right\rangle} = \sqrt{\langle x, x+y^2 \rangle} = \sqrt{\langle x, y^2 \rangle}$$
This shows that β_{r_1} and β_{r_2} are not $\mathbb{A}^1$-chain homotopic (when lifted to X_γ), but they are 1-ghost homotopic because $S^2(X) = \pi_0^{\mathbb{A}^1}(X)$
$\Rightarrow S(X)(U) \xrightarrow[\times]{} S^2(X)(U)$. □

We still have to show the characterization of chain homotopy classes mentioned before.
We are only gonna do the second part:

Prop. Fix $r_0 \in \mathfrak{m} \setminus \{0\}$, and $r_1, r_2 \in R \setminus R^\times$ such that $r_1 | r_0$, $r_2 | r_0$, $r_0 \nmid r_1$, $r_0 \nmid r_2$. Then

$$\beta_{r_1} \underset{X(r_0)}{\sim} \beta_{r_2} \iff \langle r_1 \rangle = \langle r_2 \rangle \text{ and } \frac{r_1}{r_2} - 1 \in \sqrt{\langle r_1 \rangle} + \sqrt{\left\langle \frac{r_0}{r_1} \right\rangle}$$

Proof ($\Rightarrow$) $\langle r_1 \rangle = \langle r_2 \rangle$ see lemma 4.8.
WLOG β_{r_1} and β_{r_2} are homotopic on $X(r_0)$.
Each homotopy $h: U \times \mathbb{A}^1 \to \mathbb{P}^1_U$ * it is given by an invertible sheaf on $\mathbb{A}^1_U$ and two generating sections. But U is smooth Henselian local, hence the invertible sheaf is isom. to $\mathcal{O}_{\mathbb{A}^1_U}$, hence h is given by two polynomials $p, q \in R[T]$ s.t. $\langle p, q \rangle = R[T]$. * between β_{r_1} and β_{r_2}

Note If $D(q) := \mathbb{A}^1_U \setminus V(q)$ then $h(D(q)) \subseteq Spec(R[Y/Z])$
and $\left[h|_{D(q)}: Spec(R[T][\tfrac{1}{q}]) \to Spec(R[Y/Z]), \; p/q \leftarrow\!\shortmid Y/Z \right]$

Therefore $\frac{p(0)}{q(0)} = r_1$, $\frac{p(1)}{q(1)} = r_2$.

$\Rightarrow r_1 | p(0) \notin R^\times \Rightarrow q(0) \in R^\times$, since $\langle p, q \rangle = R[T] \Rightarrow \langle p(0), q(0) \rangle = R$

Since h lifts to $X(r_0)$, we know that the ideal sheaf $h^*(I(r_0))$ is locally principal.

Since $R[T][\frac{1}{q}]$ is a UFD, we have that

$$h^*(I(r_0))(D(q)) = \langle r_0, p/q \rangle_{R[T][\frac{1}{q}]} = \langle r_1 \rangle_{R[T, \frac{1}{q}]}$$

which means that

$$p \in r_1 \cdot R[T, \tfrac{1}{q}] \cap R[T] = \bigcup_{n \geq 0} (r_1 : q^n)$$

But $(r_1 : q^n) = \langle r_1 \rangle$ for each $n \geq 0$.

└ By induction on n:
$\boxed{n=0}$ Nothing to prove
$\boxed{n \Rightarrow n+1}$ $f \in (r_1 : q^n)$, $n > 0 \Rightarrow \boxed{f q^n = r_1 g}$ for some $g \in R[T] \Rightarrow q^n | r_1 g \Rightarrow q^n \,\Big|\, \frac{r_1 g}{cont(r_1 g)} = \frac{g}{cont(g)}$
$\Rightarrow q^n | g \Rightarrow r_1 | f \Rightarrow f \in \langle r_1 \rangle$.

Hence $p \in (r_1) \Rightarrow p = r_1 \cdot p'$.

Now, $R[T] = \langle p, q \rangle = \langle r_1 p', q \rangle \Rightarrow q \in \left(\frac{R[T]}{r_1}\right)^\times$

Moreover, $\langle r_1 \rangle_{R[T, \frac{1}{q}]} = \langle r_0, \frac{r_1 p'}{q} \rangle_{R[T, \frac{1}{q}]}$

$\Rightarrow \left\langle \frac{r_0}{r_1}, \frac{p'}{q} \right\rangle_{R[T, \frac{1}{q}]} = R[T, \tfrac{1}{q}]$

$\Rightarrow \exists n \geq 0$ s.t. $q^n \in \left\langle \frac{r_0}{r_1}, p' \right\rangle_{R[T]}$.

Hence $R[T] = \left\langle \frac{r_0}{r_1}, p' \right\rangle$, because

$$R[T] = \langle p, q \rangle_{R[T]} = \langle p^n, q^n \rangle_{R[T]} = \langle (p')^n, q^n \rangle_{R[T]} \subseteq \left\langle \frac{r_0}{r_1}, p' \right\rangle$$

Therefore, if $q(T) = \sum q_j T^j$ and $p'(T) = \sum p'_j T^j$ then $q_0, p'_0 \in R^\times$ and $q_j \in \sqrt{\langle r_1 \rangle}$, $p'_j \in \sqrt{\left\langle \frac{r_0}{r_1} \right\rangle}$ for each $j \geq 1$.

Since $\frac{p(1)}{q(1)} = r_2 = \underbrace{r_1 \cdot \frac{p'(1)}{q(1)} = \boxed{r_1} \left(\frac{u_1 + t_1}{u_2 + t_2} \right)}$

where $u_1, u_2 \in R^\times$ and $t_1 \in \sqrt{\left\langle \frac{r_0}{r_1} \right\rangle}$, $t_2 \in \sqrt{\langle r_1 \rangle}$.

Since $\frac{p(0)}{q(0)} = r_1 \Rightarrow u_1 = u_2 \Rightarrow$ WLOG $u_1 = u_2 = 1$

$$\Rightarrow \boxed{\frac{r_2}{r_1} - 1} = \frac{1 + t_1}{1 + t_2} - 1 = t_1 + t_3 + t_1 t_3 \in \underbrace{\sqrt{\left\langle \frac{r_0}{r_1} \right\rangle} + \sqrt{\langle r_1 \rangle}}$$

where $t_3 = -\frac{t_2}{1 + t_2} \in \sqrt{\langle r_1 \rangle}$.

($\Leftarrow$) $\frac{r_1}{r_2} \in R^\times$ and $\frac{r_1}{r_2} - 1 \in \sqrt{\langle r_1 \rangle} + \sqrt{\left\langle \frac{r_0}{r_1} \right\rangle}$.

Note $\sqrt{\langle r_1 \rangle} = \langle s_1 \rangle$ and $\sqrt{\left\langle \frac{r_0}{r_1} \right\rangle} = \langle s' \rangle$, where $s_1, s' \in R$ are the squarefree parts of r_1 and r_0/r_1.

Then $\frac{r_1}{r_2} - 1 = s_1 \delta_1 + s' \delta'$ for some $\delta_1, \delta' \in R$.

Set $r_3 = r_1(1 + s_1 \delta_1)$. Then, we have the homotopy

$$h_1: \mathbb{A}^1_U \to Spec(R[Y/Z]) \hookrightarrow \mathbb{P}^1_U$$
$$f_1(T) \leftarrow\!\shortmid Y/Z$$

where $f_1(T) := \frac{r_3}{1 + s_1 \delta_1 T}$.

This is well-defined because

$$\langle r_3, 1 + s_1 \delta_1 T \rangle = R[T]$$

since $s_1 \delta_1 \in \sqrt{\langle r_1 \rangle} \subseteq \sqrt{\langle r_3 \rangle}$.

Then $h_1: \beta_{r_3} \sim \beta_{r_1}$ because $f_1(0) = r_3$, and $f_1(1) = \frac{r_3}{1 + s_1 \delta_1} = r_1$. Moreover, h_1 lifts to $X(r_0)$, i.e. $h^*(I(r_0))$ is locally principal.

Indeed:

$$h_1^*(I(r_0))_x = \left\langle r_0, \frac{r_3}{1 + s_1 \delta_1 T} \right\rangle_x \subseteq R[T]_x$$

is principal for each $x \in \mathbb{A}^1_U$ s.t. $(1 + s_1 \delta_1 T)|_x \in R[T]_x^\times$, since $r_3 | r_0$. If this does not happen, then $s_1|_x, \delta_1|_x, T|_x \in R[T]_x^\times$, hence

$r_1 \in R[T]_x^\times \Rightarrow r_3 = r_1(1 + s_1 \delta_1) \in R[T]_x^\times$
$\Rightarrow h_1^*(I(r_0))_x = R[T]_x$.
$\Rightarrow h_1$ lifts to $X(r_0)$.

Similarly, $h_2: \mathbb{A}^1_U \to Spec(R[Y/Z]) \hookrightarrow \mathbb{P}^1_U$
$$f_2(T) \leftarrow\!\shortmid Y/Z$$
where $f_2(T) = r_3 \left(1 + \frac{s' \delta' T}{1 + s_1 \delta_1} \right)$, relates β_{r_3} to β_{r_2} and lifts to $X(r_0)$.