2023.05.25 - TALK HQDIAG



S: scheme, $\mathrm{Spc}(S) := \mathrm{Shv}^{\mathrm{spc}}_{\mathrm{Nis}}(\mathrm{Sm}_S)$, $\mathcal{M}_1(S) := \mathrm{Shv}^{\mathrm{Sp}}_{\mathrm{Nis}}(\mathrm{Sm}_S)$

$$\begin{tikzcd} \mathrm{Spc}(S) \arrow[r, shift left, "\mathrm{Loc}_{\mathbb{A}^1}"] \arrow[d, shift right, "\Sigma^\infty"'] \arrow[r, phantom, "\perp"] & \mathcal{H}(S) \arrow[l, shift left] \arrow[d, shift right] \\ \mathrm{Ho}(\mathcal{M}_1(S)) \arrow[u, shift right, "\Omega^\infty"'] \arrow[r, shift left, "\mathrm{loc}_{\mathbb{A}^1}"] \arrow[r, phantom, "\perp"] & \mathrm{SH}_s(S) \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd}$$

$\mathcal{H}(S)$ ] unstable world

$\mathrm{SH}_s(S)$ ] stable world

[ $E \in \mathrm{Ho}(\mathcal{M}_1(S))$, $n \in \mathbb{Z}$ $\rightsquigarrow$ $h_n(E) \in \mathrm{Shv}^{\mathrm{Ab}}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ is the sheafification of $U \mapsto \pi_n(R\Gamma(U,E))$

**Def** $E \in \mathrm{Ho}(\mathcal{M}_1(S))$ is $n$-connected if $h_m(E) = 0$, $\forall m \leq n$.

**Conj.** ($\mathbb{A}^1$-connectivity) $\mathrm{loc}_{\mathbb{A}^1} : \mathrm{Ho}(\mathcal{M}_1(S)) \to \mathrm{SH}_s(S)$ preserves (-1)-connected objects. $S = \mathrm{Spec}(k)$

**Conj.** ($\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}$) $\forall X \in \mathrm{Spc}(S)$, $\pi_0^{\mathbb{A}^1}(X) := \pi_0(\mathrm{loc}_{\mathbb{A}^1}(X))$ is $\mathbb{A}^1$-invariant.

**Thm** (Ayoub) Both conjectures are false.

$\boxed{\mathbb{A}^1\text{-connectivity}}$ First, we pass to the abelian setting.

$\mathcal{M}_2(S) := \mathrm{Compl}(\mathrm{Shv}^{\mathrm{ab}}_{\mathrm{Nis}}(\mathrm{Sm}_S))$

$\mathcal{M}_3(S) := \mathrm{Compl}(\mathrm{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathrm{Sm}_S))$

2) <u>Lemma</u> If Conj. 1 holds for $S$ then the functor $\mathrm{Loc}_{\mathbb{A}^1} : \mathrm{Ho}(\mathcal{M}_i(S)) \to \mathrm{Ho}_{\mathbb{A}^1}(\mathcal{M}_i(S))$ preserves $(-1)$-connected objects, $\forall i \in \{1,2,3\}$.

<u>Proof</u> Eilenberg-MacLane objects.

Therefore, to disprove Conj. 1, it's sufficient to work with $\mathcal{M}_3(S)$, for which we have $\mathrm{Ho}_{\mathbb{A}^1}(\mathcal{M}_3(S)) = DM_{eff}(S)$.

Moreover, we will use the following characterization.

<u>Thm</u> If $S$ satisfies Conj. 1 and $E \in \mathrm{Ho}_{\mathbb{A}^1}(\mathcal{M}_3(S))$, TFAE:

(1) $E \in DM_{eff}(S)$ (2) $E_{\geq n} \in DM_{eff}(S)$, $\forall n \in \mathbb{Z}$

(3) $h_n(E)$ is strictly $\mathbb{A}^1$-invariant, $\forall n \in \mathbb{Z}$

<u>Proof</u> (3) $\Rightarrow$ (1) Omitted

(1) $\Rightarrow$ (2) WLOG $n=0$.

$$\begin{tikzcd} E_{\geq 0} \arrow[r] & \overbrace{\mathrm{Loc}_{\mathbb{A}^1}(E_{\geq 0})}^{\text{is } (-1)\text{-connected}} \arrow[r] \arrow[dr, dashed] & \mathrm{loc}_{\mathbb{A}^1}(E) \cong E \\ & & E_{\geq 0} \arrow[u] \end{tikzcd}$$

$\Rightarrow E_{\geq 0}$ is a direct factor of $\mathrm{Loc}_{\mathbb{A}^1}(E_{\geq 0}) \Rightarrow E_{\geq 0} \in DM_{eff}(S)$

(2) $\Rightarrow$ (3) $H(h_n(E))[n] \to E_{\geq n} \to E_{\geq n-1}$.

Thus, to disprove Conj. 1, we will construct $E \in DM_{eff}(S)$ s.t. $h_n(E)$ is not strictly $\mathbb{A}^1$-invariant for some $n \in \mathbb{Z}$ (we will take $n=-1$).

$k$: perfect field, $K_n^M := h_0(\mathrm{loc}_{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{G}_m,1)^{\wedge n}))$ the $n$-th Milnor K-theory sheaf, which is an $\mathbb{A}^1$-invariant Nisnevich sheaf with transfers, yielding $K_n^M \in DM_{eff}(k)$ by considering it as a complex concentrated in degree $0$.
This sheaf can be used to form a Cousin complex

$$\Big[\coprod_{U \in U_{(0)}} K_n^M(U) \to \coprod_{U^{(1)} \in U_{(1)}} K_{n-1}^M(U^{(1)}) \to \_ \to \coprod_{U^{(n)} \in U_{(n)}} K_0^M(U)\Big]$$

for each $U \in \mathrm{Sm}_S$, where $U_{(d)}$ is the set of points having codimension $d$. This yields an element $K_{S,n}^{M,!} \in DM_{eff}(S)$, which admits the alternative descriptions

$$K_{S,n}^{M,!} = \pi_S^! K_{n-d_S}^M[-d_S] = i^! \pi_W^*(K_{n+c}^M[c])$$

where
$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[d, "\pi_S"'] & W \arrow[dl, "\pi_W"] \\ \mathrm{Spec}(k) & \end{tikzcd}$$
and $i$ is an immersion of codimension $c$ with $W$ regular.

<u>Thm</u> (Ayoub) $X_{/k} \hookrightarrow \mathbb{P}^3$ surface [redacted], $X: \{x_0(x_1^3 - x_2^2 x_3) + f(x_1,x_2,x_3) = 0\}$ for $f$ homogeneous, $\deg(f) = 4$, generic. Then $h_{-1}(K_{X,1}^{M,!})$ is not strictly $\mathbb{A}^1$-invariant.

$X_{/k}$ normal surface with a single singular point $s \in X$. If $U \to X$ is a connected smooth $X$-scheme we have that

4| $h^{M_1!}_{X,1}(U) = [0 \to k(U)^\times \to \bigoplus_{u \in U_{(1)}} \mathbb{Z} \to 0]$, hence

$H^1_{Nis}(U, h^{M_1!}_{X,1}) = H_{-1}(R\Gamma(U, h^{M_1!}_{X,1})) \cong Cl(U)$, which implies that $h_{-1}(h^{M_1!}_{X,1}) = cl_X$ where $cl_X$ is the sheafification of $U \mapsto Cl(U)$. We want to show that $cl_X$ is not strictly $\mathbb{A}^1$-invariant.

<u>Lemma</u> If $U \underset{\text{smooth}}{\to} X \setminus \{g\} \hookrightarrow X$ then $cl_X(U) = 0$. Hence, if $cl_X$ is strictly $\mathbb{A}^1$-invariant, $cl_X = 1_* F$ for some $\mathbb{A}^1$-inv. sheaf with transferts $F$ on $Sm_k$.

<u>Proof</u> $U \to X \setminus \{g\}$ smooth $\Rightarrow$ $U$ regular $\Rightarrow$ [redacted] $U$ can be covered by regular local schemes $V$, for which $Cl(V) = Pic(V) = 0 \Rightarrow cl_X(U) = 0$. By the localization theorem, we have an exact triangle in $DM_{eff}(X)$:

$$Loc_{\mathbb{A}^1} j_\# j^* cl_X \to cl_X \to 1_* Loc_{\mathbb{A}^1} L 1^* cl_X \xrightarrow{+1}$$

which yields $cl_X \xrightarrow{\sim} 1_* Loc_{\mathbb{A}^1} L 1^* cl_X$ since $j^* cl_X = 0$.

Now, we fix a resolution of singularities $X' \twoheadrightarrow X$, which yields the following diagram for each smooth scheme $U \to X$. If $U$ is connected, we get the following sequence of maps

$$\begin{tikzcd}
& U_C \arrow[dl] \arrow[rr] \arrow[dd] & & U' \arrow[dl] \arrow[dd] \\
C \arrow[r] \arrow[dd, "j"] & & X' & \\
& U_1 \arrow[dl] \arrow[rr] & & U \arrow[dl] \\
\{g\} \arrow[rr, hook] & & X \arrow[from=uu, crossing over] &
\end{tikzcd}$$

$$Cl(U) \to Pic(U \setminus U_1) \simeq Pic(U' \setminus U_C) \simeq \frac{Pic(U')}{R_0(U)} \hookrightarrow \frac{Pic(U_C)}{R(U)}$$

where $R_0(U) \subseteq Pic(U')$ is the subgroup of divisors supported on $U_C$, and $R(U)$ is its image under $Pic(U') \hookrightarrow Pic(U_C)$.

<u>Rmk</u> If $C_1, \dots, C_n$ are the irreducible components of $C$, then $R_0(U) = \langle [U \times_X C_i] \rangle$. Hence, $R_0$ and $R$ are constant sheaves of finite rank.

The previous map yields a map of sheaves $\alpha : cl_X \to \iota_*(Pic_C / R)$ where $Pic_C := a_{Nis}(U \mapsto Pic(U \times_X C))$.

<u>Lemma</u> $\alpha$ is surjective

<u>Proof</u> It is sufficient to show that $Pic(U') \hookrightarrow Pic(U_C)$ is surjective for each $U \to X$ local, henselian and pro-smooth. Take a class $a \in Pic(U_C)$ of some line bundle which is relatively very ample over $U_1$, so that $a = [Z_1]$ with $Z_1 \hookrightarrow U_C$ such that $Z_1 \to U_1$ is finite and $Z_1 \subseteq U_C^{reg}$. We can assume that $Z_1$ is irreducible. Take $V \hookrightarrow U'$ to be an affine open such that $Z_1 \subseteq V$ and $V$ trivializes $a$. We can assume that $V \cap U_C$ is dense in each fiber of $U_C \to U_1$, thanks to relative ampleness. Then, we have that $Z_1 = \{f = 0\}$ in $V \cap U_C$. Lifting $f$, this yields $Z' := \{f = 0\} \hookrightarrow V$ so that $Z_1 = Z' \cap V \cap U_C$. Then, let $Z$ be the closure of $Z'$ in $U'$, which gives that $Z_1 = Z \cap V \cap U_C$. Hence, we have $(Z \cap U_C) \setminus Z_1 \subseteq U_C \setminus (U_C \cap V) \hookrightarrow U_C$ and disjoint from $Z_1$. Hence, $Z_1$ is a connected component of $Z \cap U_C$. Now, observe that $Z \to U$ is finite. Indeed, $Z \to U$ is projective since $X' \to X$ is projective, hence $U' \to U$ is also projective.

6| Now, observe that $Z \cap U_C \subseteq Z_1 \cup (U_C \setminus V)$ which implies that $Z \to U$ is finite because $U_C \setminus V$ is dense in the fibers of $U_C \to U_1$. Hence, $Z$ is a disjoint union of henselian schemes, and $Z_1 = Z'' \cap U_C$ where $Z'' \subseteq Z$ is the connected component containing $Z_1$. This gives $[Z''] \mapsto a$ under $\mathrm{Pic}(U_C') \hookrightarrow \mathrm{Pic}(U_C)$.

<u>Cor.</u> If $\mathcal{C}l_X$ is strictly $\mathbb{A}^1$-invariant, $\mathrm{Pic}_C$ is $\mathbb{A}^1$-invariant.

<u>Proof</u> $\mathcal{C}l_X$ is strictly $\mathbb{A}^1$-invariant $\Rightarrow$ $\mathcal{C}l_X = \mathcal{I}_* F$ for some $\mathbb{A}^1$-invariant sheaf with transfers $F$. Then $\alpha : \mathcal{I}_* F \twoheadrightarrow \mathcal{I}_*\left(\frac{\mathrm{Pic}_C}{R}\right)$ comes from $F \twoheadrightarrow \mathrm{Pic}_C/R$ which implies that $\mathrm{Pic}_C/R$ is $\mathbb{A}^1$-invariant, and hence that $\mathrm{Pic}_C$ is $\mathbb{A}^1$-invariant because $R$ is a constant sheaf of finite rank.

<u>Proof</u> (of the theorem) $X : \{x_0(x_1^3 - x_2^2 x_3) + f(x_1, x_2, x_3) = 0\}$ is smooth outside $s = [1:0:0:0]$. We can resolve the singularity blowing up $s$, which gives $C \cong \{x_1^3 - x_2^2 x_3\}$ so that $\mathrm{Pic}_C = \mathbb{Z} \oplus \mathbb{G}_a$.

<u>Rmk</u> This implies that each scheme $S/_k$ with $\dim(S) \geq 2$ does not have the $\mathbb{A}^1$-connectivity property. Indeed:

- if $\dim(S) = 2$, we can assume WLOG that $S = \mathbb{A}^2_k$ by Noether normalization. In this case, $h_{-1}(\pi_* K^{M,!}_{U,1})$ is not strictly $\mathbb{A}^1$-invariant, where $U \to \mathbb{A}^2$ is a finite map and $U \hookrightarrow X$ is an affine neighbourhood of the singular point;
- if $\dim(S) \geq 3$, we can use the fact that if the $\mathbb{A}^1$-connectivity property holds for $S$ then it also

holds for every closed subscheme $T \overset{i}{\hookrightarrow} S$. Indeed, in this case $Ri_* = i_*$, hence $Ri_*$ preserves $(-1)$-connected objects. Moreover, $\mathrm{Loc}^S_{\mathbb{A}^1} Ri_* \to Ri_* \mathrm{Loc}^T_{\mathbb{A}^1}$ is a natural equivalence. Therefore, $Ri_* \mathrm{Loc}^T_{\mathbb{A}^1}$ preserves [redacted] $(-1)$-connectedness [redacted], which can only happen when $\mathrm{Loc}^T_{\mathbb{A}^1}$ does [redacted] so.

$\boxed{\mathbb{A}^1\text{-invariance of } \pi_0^{\mathbb{A}^1}}$

**Def** $X \in Sm_k$ is $\mathbb{A}^1$-discrete if for each field extension $K/k$, each map $\mathbb{A}^1_K \to X$ factors through a $K$-point $\mathrm{Spec}(K) \to X$.

Ex. Abelian varieties and powers of curves of genus $\geq 1$ are $\mathbb{A}^1$-discrete.

Let $X$ be an $\mathbb{A}^1$-discrete smooth variety $/k$. Then, we can always find an integral surface $Y \hookrightarrow X$ with $o \in Y(k)$, and a Zariski neighborhood $o \in N \hookrightarrow Y$ which is étale over $o \in S := \{ x_0(x_1^3 - x_2^2 x_3) + f(x_1, x_2, x_3) = 0 \}$. Then, as before, we have a complex of abelian groups $K^{M!}_{S,1}$ concentrated in homological degrees zero and one, such that $K^{M!}_{S,1}(U) = \left[ k(U)^\times \to \bigoplus_{v \in U_{(1)}} \mathbb{Z} \right]$ for each $U \in Sm_S$. This has an associated Eilenberg-MacLane space $K^{M!}_{S,1} \in \mathcal{H}(S)$ which induces $M := i_* e^* K^{M!}_{S,1} \in \mathcal{H}(X)$ such that $\pi_0^{\mathbb{A}^1}(M)$ is not $\mathbb{A}^1$-invariant.

To conclude, we will show that the formalism is sufficiently flexible to allow us to pass from $\mathcal{H}(X)$ to $\mathcal{H}(k)$, using a functor $\Phi_X : \mathcal{H}(X) \to PSh$ defined by

 $\Phi_X(M) := \left(U \mapsto \coprod_{s \in \mathrm{Hom}(U,X)} \Gamma(U, s^*M)\right)$. This is in fact a functor of $\infty$-categories, thanks to the alternative definition

$$\Phi_X : \mathcal{H}(X) \underset{(+)}{\xrightarrow{\sim}} \mathrm{Sect}^{\mathrm{cocart}}\left(\int_{(Sm_k(X))^{op}} \mathcal{H}(U) \Big/ Sm_k(X)^{op}\right) \xrightarrow{\Gamma} PSh(Sm_k(X)) \xrightarrow{f_{X,\#}} PSh(Sm_k)$$

where $(+)$ comes from the Grothendieck construction, $\Gamma$ is the global sections functor, and $f_{X,\#}$ is the left Kan extension along $f_X : Sm_{k/X} \to Sm_k$.

<u>Rmk</u> $Loc_{\mathbb{A}^1}(a_{Nis}(\Phi_X(M))) \cong p_\#(M)$ where $p : X \to Spec(k)$ and $p_\# \dashv p^* : \mathcal{H}(k) \to \mathcal{H}(X)$.

<u>Prop.</u> $X$ proper and $\mathbb{A}^1$-discrete, $M \in \mathcal{H}(X)$ $\Rightarrow$ $\Phi_X(M) \in \mathcal{H}(k)$

<u>Proof</u> First of all, we check that $\Phi_X(M)$ is a Nisnevich sheaf. Indeed:

- $\Phi_X(M)(\emptyset) = *$ since $\mathcal{H}(\emptyset)$ is the final category.
- $\Phi_X(M)(U_1 \sqcup U_2) = \Phi_X(M)(U_1) \times \Phi_X(M)(U_2)$ since
$$\begin{cases} \mathrm{Hom}(U_1 \sqcup U_2, X) = \mathrm{Hom}(U_1, X) \times \mathrm{Hom}(U_2, X) \\ \Gamma(U_1 \sqcup U_2, s^*M) = \Gamma(U_1, s_1^*M) \times \Gamma(U_2, s_2^*M) \end{cases}$$
- for each Nisnevich square
$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "e'"'] & V' \arrow[d, "e"] \arrow[r] & {} \arrow[d, "s"] \\ U \arrow[r, hook, "j"'] & V \arrow[r] & {} \end{tikzcd}$$
we have that
$$\begin{tikzcd} \Phi_X(M)(U') & \Phi_X(M)(V') \arrow[l] \arrow[dl, phantom, "(\mathrm{I})\ \lrcorner"] \\ \Phi_X(M)(U) \arrow[u] & \Phi_X(M)(V) \arrow[l] \arrow[u] \end{tikzcd}$$

Indeed, without loss of generality we can assume that $U$ and $V'$ are connected, and that $j$ and $j'$ have dense images. Then, $\mathrm{Hom}(V',X) = \mathrm{Hom}(U',X)$ and $\mathrm{Hom}(V,X) = \mathrm{Hom}(U,X)$ because rational maps to a proper, $\mathbb{A}^1$-discrete variety are everywhere defined. Hence, (I) becomes

$$\begin{tikzcd}
\coprod_{V' \xrightarrow{t'} X} \Gamma(U', (t')^*M) & \coprod_{V' \xrightarrow{t'} X} \Gamma(V', (t')^*M) \arrow[l] \\
\coprod_{V \xrightarrow[t]{} X} \Gamma(U, t^*M) \arrow[u] \arrow[ur, phantom, "(\mathrm{II})"] & \coprod_{V \xrightarrow[t]{} X} \Gamma(V, t^*M) \arrow[l] \arrow[u]
\end{tikzcd}$$

Now, since $e$ is étale, $\mathrm{Hom}(V,X) \to \mathrm{Hom}(V',X)$ is injective. Hence (II) is Cartesian iff the following square

$$\begin{tikzcd}
\coprod_{V \xrightarrow[t]{} X} \Gamma(U', t^*M) & \coprod_{V \xrightarrow[t]{} X} \Gamma(V', t^*M) \arrow[l] \\
\coprod_{V \xrightarrow[t]{} X} \Gamma(U, t^*M) \arrow[u] & \coprod_{V \xrightarrow[t]{} X} \Gamma(V, t^*M) \arrow[l] \arrow[u]
\end{tikzcd}$$

is Cartesian, which happens iff each of the squares

$$\begin{tikzcd}
\Gamma(U', t^*M) & \Gamma(V', t^*M) \arrow[l] \\
\Gamma(U, t^*M) \arrow[u] & \Gamma(V, t^*M) \arrow[l] \arrow[u]
\end{tikzcd}$$

is Cartesian. This holds true because $t^*M$ is a Nisnevich sheaf.

10| Similarly, one can show that $\Phi_X(M)$ is $\mathbb{A}^1$-invariant. Indeed, we have that:

$$\begin{array}{ccc} \Phi_X(M)(U) & \to & \Phi_X(M)(U\times\mathbb{A}^1) \\ \wr\| & & \wr\| \\ \coprod\limits_{s:U\to X}\Gamma(U,s^*M) & \to & \coprod\limits_{\underbrace{U\times\mathbb{A}^1\xrightarrow{s'} X}_{=U\xrightarrow{s}X}} \Gamma(U\times\mathbb{A}^1, \underbrace{(s')^*M}_{s^*M}) \end{array}$$

which implies that $\Phi_X(M)(U) \xrightarrow{\sim} \Phi_X(M)(U\times\mathbb{A}^1)$ because $\Gamma(U,s^*M)\xrightarrow{\sim}\Gamma(U\times\mathbb{A}^1,s^*M)$.

<u>Prop.</u> $X$ proper and $\mathbb{A}^1$-discrete, then

$$\pi_0^{\mathbb{A}^1}(\Phi_X(M))(U) = \coprod_{U\xrightarrow{s}X}\Gamma(U,\pi_0^{\mathbb{A}^1}(s^*M))$$

<u>Proof</u> Since $\Phi_X(M)\in\mathcal{H}(k)$, we have that $F := \pi_0^{\mathbb{A}^1}(\Phi_X(M)) = \pi_0(\Phi_X(M$ is the sheafification of $U\mapsto \coprod\limits_{U\xrightarrow{s}X}\pi_0\Gamma(U,s^*M)$.

On the other hand, we have a presheaf $G\in PSh(Sm_k)$ defined by $G(U)=\coprod\limits_{U\xrightarrow{s}X}\Gamma(U,\pi_0^{\mathbb{A}^1}(s^*M))$ which is actually a Nisnevich sheaf, and we have a map $F\to G$. This is an isomorphism on stalks, because if $W/k$ is henselian and essentially smooth then we have that the map

$$\pi_0\Gamma(W;s^*M)\xrightarrow{\sim}\Gamma(W;\pi_0(s^*M))=\Gamma(W;\pi_0^{\mathbb{A}^1}(s^*M))$$

is obviously an isomorphism, because $\pi_0(s^*M)$ is the sheafification of $V \mapsto \pi_0 \Gamma(V; s^*M)$.

<u>Cor.</u> $X$ proper and $\mathbb{A}^1$-discrete, $\pi_0^{\mathbb{A}^1}(\Phi_X(M))$ is $\mathbb{A}^1$-invariant iff $\pi_0^{\mathbb{A}^1}(s^*M)$ is $\mathbb{A}^1$-invariant for each $U \xrightarrow{s} X$.

<u>Proof</u>

$$\begin{tikzcd}
\pi_0^{\mathbb{A}^1}(\Phi_X(M))(U) \arrow[r, "\sim"] \arrow[d, phantom, "\wr\|"] & \pi_0^{\mathbb{A}^1}(\Phi_X(M))(U \times \mathbb{A}^1) \arrow[d, phantom, "\wr\|"] \\
\coprod\limits_{U \xrightarrow{s} X} \Gamma(U, \pi_0^{\mathbb{A}^1}(s^*M)) \arrow[r, "\sim"] & \coprod\limits_{\underbrace{\{U \times \mathbb{A}^1 \xrightarrow{t} X\}}_{\sim \{U \xrightarrow{s} X\}}} \Gamma(U \times \mathbb{A}^1, \pi_0^{\mathbb{A}^1}(t^*M))
\end{tikzcd}$$

$\Longleftrightarrow \forall s : U \to X$, $\Gamma(U, \pi_0^{\mathbb{A}^1}(s^*M)) \xrightarrow{\sim} \Gamma(U \times \mathbb{A}^1, \pi_0^{\mathbb{A}^1}(s^*M))$

$\Longleftrightarrow \pi_0^{\mathbb{A}^1}(s^*M) \in \mathcal{H}(U)$, $\forall\, U \xrightarrow{s} X$.