BONUS We have used the following fact.

Cor. $X \overset{f}{\dashrightarrow} Y$ rational, X ~~smooth~~ smooth, Y proper without rational curves $\Rightarrow$ [redacted] f is defined everywhere.

Proof Let $Z \subseteq X \times Y$ be the graph of f, and $Z \overset{p}{\to} X$ be the first projection, which is proper birational. If $\mathrm{Exc}(p) \neq \emptyset$, it contains a rational curve contracted by p, which has to be contracted also by $Z \to Y$. But this is absurd! Hence $\mathrm{Exc}(p) = \emptyset$ and f is everywhere defined.

The proof uses crucially the following result.

Prop. $X \xrightarrow{p} Y$ birational, Y smooth $\Rightarrow$ Each component of $Exc(p)$ is ruled, i.e. birational to $\mathbb{P}^1 \times Z$ for some Z.

Proof First, we can perform several reductions, assuming that

- Y is normal (replace Y by normalization)
- $E := Exc(p)$ and Y are smooth irreducible (shrink Y)
- $\overline{p(E)}$ is smooth of codimension ≥ 2 (shrink X and Y), because each component of E has codim. 1.

Then, consider the blow-up $Y_1 \to Y$ of $\overline{p(E)}$. Since p is birational, we have $p : X \xrightarrow[p_1]{} Y_1 \xrightarrow[\varepsilon_1]{} Y$ where $\overline{p_1(E)} \subseteq Exc(\varepsilon_1)$. If the codimension of $\overline{p_1(E)}$ is ≥ 2, then $E \subseteq Exc(p_1)$ and we may get a factorization $p : X \xrightarrow[p_n]{} Y_n \xrightarrow[\varepsilon_n]{} Y_{n-1} \to \dots \to Y_1 \xrightarrow[\varepsilon_1]{} Y$ where the codimension of $\overline{p_i(E)}$ is ≥ 2 for each $i \leq n-1$. Note that this process terminates, i.e. $\exists n \geq 1$ s.t. $\overline{p_n(E)}$ is a divisor. Indeed, $K_{Y_n} = (\varepsilon_1 \circ \dots \circ \varepsilon_n)^* K_Y + c_1 E_1 + \dots + c_n E_n$ where $E_i = Exc(\varepsilon_i)$ and $c_i = \mathrm{codim}(\overline{p_{i-1}(E)}) - 1$.

Moreover, $p_i^* E_i - E \geq 0$ since $p_n(E) \subseteq E_n$, and since p_n is birational, $p_n^*(\mathcal{O}(K_{X_n}))$ is a subsheaf of $\mathcal{O}_X(K_X)$, and so is $p^*(\mathcal{O}(K_Y)) + (c_1 + \dots + c_n)\mathcal{O}(E)$.

Since X is Noetherian and $\mathcal{O}(K_X)$ is coherent, the process must terminate, i.e. $c_n = 0$. Then p_n induces a birational iso between E and E_n, and the latter is ruled by construction.