Two conjectures of Morel

S: scheme, $Spc(S) := Shv^{spc}_{Nis}(Sm_S)$,

$M_1(S) := Spect(Shv^{Sets}_{Nis}(Sm_S))$

unstable

$$\begin{tikzcd} Spc(S) \arrow[r, shift left, "loc_{\mathbb{A}^1}"] \arrow[d, shift right, "\Sigma^\infty"'] \arrow[r, phantom, "\perp"] & \mathcal{H}(S) \arrow[l, shift left] \arrow[d, shift right] \\ Ho(M_1(S)) \arrow[r, shift left, "loc_{\mathbb{A}^1}"] \arrow[u, shift right, "\Omega^\infty"'] \arrow[r, phantom, "\perp"] & SH_1(S) \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd}$$

$\mathcal{H}(S)$: $\mathbb{A}^1$-invariant

stable

Def $E \in Ho(M_1(S))$, $h_n(E) \in Shv^{Ab}_{Nis}(Sm_S)$ is the sheafification of $U \mapsto \pi_n(R\Gamma(U, E))$

Def $E \in Ho(M_1(S))$ is n-connected if $h_m(E) = 0$ $\forall m \leq n$.

Conj.^{1} (Morel, $\mathbb{A}^1$-connectivity) $loc_{\mathbb{A}^1}: Ho(M_1(S)) \to SH_1(S)$ preserves (-1)-connected objects, $\forall S$ regular.

Conj.^{2} (Morel, $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}$) $S = Spec(k)$, k perfect field $\Rightarrow \forall X \in Spc(S)$, $\pi_0^{\mathbb{A}^1}(X) := \pi_0(loc_{\mathbb{A}^1}(X))$ is $\mathbb{A}^1$-invariant.

Thm (Ayoub) Both conjectures are false!

$\boxed{\mathbb{A}^1\text{- connectivity}}$ $M_2(S) := Cpl(Shv^{ab}_{Nis}(Sm_S))$

$M_3(S) := Cpl(Shv^{tr}_{Nis}(Sm_S))$

Lemma If Conj. 1 holds for S, then the functors

$loc_{\mathbb{A}^1}: Ho(M_i(S)) \longrightarrow Ho_{\mathbb{A}^1}(M_i(S))$

preserve (-1)-connected objects, $\forall i \in \{1,2,3\}$.

Proof Eilenberg-MacLane objects (no adjunctions)

From now on, we are gonna work with $M_3(S)$, and

$DM_{eff}(S) := Ho_{\mathbb{A}^1}(M_3(S))$.

Thm (Morel) If S satisfies conj. 1, and $E \in Ho(M_3(S))$ then TFAE:

(1) $E \in DM_{eff}(S)$ (2) $E_{\geq n} \in DM_{eff}(S)$, $\forall n \in \mathbb{Z}$

(3) $h_n(E)$ is strictly $\mathbb{A}^1$-invariant, $\forall n \in \mathbb{Z}$

Proof (3) $\Rightarrow$ (1) omitted

(1) $\Rightarrow$ (2) WLOG $n = 0$ $\Rightarrow$ $E_{\geq 0} \to \overbrace{loc_{\mathbb{A}^1}(E_{\geq 0})}^{(-1)\text{-connected}} \to loc_{\mathbb{A}^1}(E) \overset{\sim}{=} E \leftarrow E_{\geq 0}$

$\Rightarrow E_{\geq 0}$ is a direct factor of $loc_{\mathbb{A}^1}(E_{\geq 0}) \Rightarrow E_{\geq 0} \in DM_{eff}(S)$

(2) $\Rightarrow$ (3) $\underbrace{H(h_n(E))[n]}_{\in DM_{eff}(S)} \to E_{\geq n} \to E_{\geq n-1}$ ($E_{\geq n}$, $E_{\geq n-1}$ $\in DM_{eff}(S)$)

$\Rightarrow h_n(E)$ is strictly $\mathbb{A}^1$-invariant.

Rmk Alternative way with hypercohomology s.s.

Thus, to disprove Conj. 1 it's sufficient to construct $E \in DM_{eff}(S)$ s.t. $h_n(E)$ is not strictly $\mathbb{A}^1$-invariant for some $n \in \mathbb{Z}$ (in fact, we will take $n = -1$).

k: perfect field, $n \in \mathbb{Z}$:

Def $K_n^M := h_0(loc_{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{G}_m, 1)^{\wedge n})))$ [unramified sheaf associated to Milnor K-theory]

This yields $K_n^M \in DM_{eff}(k)$ by seeing the sheaf K_n^M as a complex in degree zero.

$F \rightsquigarrow F_{-n} := \underline{Hom}((\mathbb{G}_m, 1)^{\wedge n}, F)$

$\rightsquigarrow (K_n^M)_{-1} = K^M_{n-1}$.

Def $\underbrace{S \xrightarrow{\pi_S} Spec(k)}_{d_S = \dim(S)}$, $K^{M,!}_{S,n} := \pi_S^! K^M_{n-d_S}[-d_S] \in DM_{eff}(S)$

- $\underbrace{S \overset{i}{\hookrightarrow} W}_{\text{codim } c}$, W regular $\Rightarrow$ $\underbrace{K^{M,!}_{S,n} = i^! \pi_W^*(K^M_{n+c}[c])}$
- $U \to S$ smooth $\Rightarrow K^{M,!}_{S,n}(U) = \left[\coprod_{u \in U_{(0)}} K^M_n(u) \to \coprod_{u^{(1)} \in U_{(1)}} K^M_{n-1}(u^{(1)}) \to \ldots \to \coprod_{u^{(n)} \in U_{(n)}} K^M_0(u^{(n)})\right]$ where $U_{(j)}$ denotes the set of points of U having codimension j, where $j \in \{0, \ldots, n\}$.

Thm (Ayoub) $X_{/k}$ surface, $X: \{x_0(x_1^3 - x_2^2 x_3) + f(x_1, x_2, x_3) = 0\} \hookrightarrow \mathbb{P}^3$ for f homogeneous, $\deg(f) = 4$, "generic". Then $h_{-1}(K^{M,!}_{X,1})$ is not strictly $\mathbb{A}^1$-invariant.

$X_{/k}$ normal surface, with a single singular point $o \in X(k)$.

$U \to X$ connected, smooth X-scheme, then

$K^{M,!}_{X,1}(U) = [0 \to k(U)^\times \xrightarrow{div} \overbrace{\bigoplus_{u \in U_{(1)}} \mathbb{Z}} \to 0]$, hence

$H^1_{Nis}(U, K^{M,!}_{X,1}) = \underbrace{H_{-1}(R\Gamma(U, K^{M,!}_{X,1}))} \cong Cl(U)$

$\Rightarrow h_{-1}(K^{M,!}_{X,1}) =: Cl_X$ is the sheafification of $U \mapsto Cl(U)$.

Lemma If $U \xrightarrow[\text{smooth}]{} X \setminus \{z\} \hookrightarrow X \Rightarrow cl_X(U) = 0$.

Hence if cl_X is strictly $\mathbb{A}^1$-invariant, $cl_X = i_* F$ where F is some $\mathbb{A}^1$-invariant $\underset{\text{couple of sheaves}}{\wedge}$ with transfers on Sm_k.

Proof $U \xrightarrow[\text{smooth}]{} X \setminus \{z\} \Rightarrow U$ regular $\Rightarrow U$ can be covered (in the Nis. topology) by regular local schemes V, for which $Cl(V) = Pic(V) = 0 \Rightarrow cl_X(U) = 0$.

For the second part, look at the localization triangle:

$$loc_{\mathbb{A}^1} j_\# \overbrace{j^* cl_X}^{=0} \to \boxed{loc_{\mathbb{A}^1} cl_X = cl_X} \to i_* \overset{h_0}{\vee}(\underbrace{loc_{\mathbb{A}^1} i^* cl_X}_{=:F}) \xrightarrow{+1}$$

$\Rightarrow cl_X \cong i_* F$.

Take a resolution of singularities $X' \twoheadrightarrow X$, yielding the cube on the left for each $U \to X$ smooth.

$$\begin{tikzcd}
 & U_C \arrow[rr] \arrow[dl] \arrow[dd] & & U' \arrow[dl] \arrow[dd] \\
C \arrow[rr, hook] \arrow[dd] & & X' \arrow[dd] & \\
 & U_z \arrow[rr] \arrow[dl] & & U \arrow[dl, "\text{smooth}"] \\
\{z\} \arrow[rr, hook] & & X &
\end{tikzcd}$$

If U is connected, we get the following sequence of maps:

$$Cl(U) \to Pic(U \setminus U_z) \cong Pic(U' \setminus U_C) \cong \frac{Pic(U')}{R_0(U)} \xrightarrow{\alpha} \frac{Pic(U_C)}{R(U)} \quad (\dagger)$$

$R_0(U) \subseteq Pic(U')$: the subgroup of divisors supported on U_C

$R(U) \subseteq Pic(U_C)$ is the image of $R_0(U)$ under $Pic(U') \to Pic(U_C)$.

Rmk If $C_1, \dots, C_n$ are the irr. comp. of C, then $R_0(U) = \langle [U \times C_i] \rangle \Rightarrow R_0$ is a constant sheaf of finite rank.

$(\dagger)$ yields a map of sheaves $\alpha : cl_X \to i_*(Pic_C / R)$ where $Pic_C := a_{Nis}(U \mapsto Pic(U \times C))$.

Lemma α is surjective.

Proof We only have to show that $Pic(U') \to Pic(U_C)$ is surjective, for each $U \to X$ local, henselian and pro-smooth. Take $a \in Pic(U_C)$ associated to a line bundle which is very ample relatively to $U_C \to U_z$. Then $a = [Z_z]$ where $Z_z \hookrightarrow U_C$ s.t. $Z_z \to U_z$ finite, and $Z_z \subseteq U_C^{reg}$.

WLOG: Z_z is irreducible. Take $V \hookrightarrow U'$ affine open s.t. $Z_z \subseteq V$ and V trivializes a. Thanks to ampleness, we can assume that $V \cap U_C$ is dense in each fiber $U_C \to U_z$.

$\Rightarrow Z_z = \{f = 0\}$ in $V \cap U_C$. Lifting f, we obtain $Z' := \{f = 0\} \hookrightarrow V$ s.t. $Z_z = Z' \cap U_C$.

Take Z to be the closure of Z' in U', so that $Z_z = Z \cap V \cap U_C$. Hence $(Z \cap U_C) \setminus Z_z \subseteq U_C \setminus (U_C \cap V)$ with $U_C \setminus (U_C \cap V)$ closed and disjoint from Z_z

$\Rightarrow Z_z$ is a connected component of $Z \cap U_C$.

$\Rightarrow Z_z = Z'' \cap U_C$ where $Z'' \subseteq Z$ is the connected component containing Z_z. This is because $Z \to U$ is finite, hence Z is a disjoint union of henselian schemes.

Cor. If cl_X is strictly $\mathbb{A}^1$-invariant, then Pic_C is $\mathbb{A}^1$-invariant

Proof $cl_X = i_* F \overset{\alpha}{\twoheadrightarrow} i_*\left(\frac{Pic_C}{R}\right) \Rightarrow \alpha$ has to come from $\underset{DM^{eff}}{\overset{\cap}{\boxed{F}}} \twoheadrightarrow \frac{Pic_C}{R} \Rightarrow \frac{Pic_C}{R}$ is $\mathbb{A}^1$-invariant

$\underset{\substack{\Uparrow \\ R \text{ constant} \\ \text{of finite rank}}}{\Rightarrow} Pic_C$ is $\mathbb{A}^1$-invariant

Proof (of the theorem) $X : \{x_0(x_1^3 - x_2^2 x_3) + f(x_1, x_2, x_3) = 0\}$ is smooth outside $z = [1 : 0 : 0 : 0]$, and $C \cong \{x_1^3 - x_2^2 x_3 = 0\}$ so that $Pic_C = \mathbb{Z} \oplus \mathbb{G}_a$ is not $\mathbb{A}^1$-invariant.

Rmk For every $S \to \mathrm{Spec}(k)$ with $\dim(S) \geq 2$, the $\mathbb{A}^1$-conn. conj. fails.

$\boxed{\text{$\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}$}}$

X: smooth variety $/k$, $\dim(X) \geq 3$

$Y \hookrightarrow X$: integral surface, $o \in Y(k)$

$$\begin{tikzcd} o \arrow[d] & \in & N \arrow[r, hook, "\text{Zar.}"] \arrow[d, "e"', "\text{étale}"] \arrow[rr, bend right, "i"'] & Y \arrow[r, hook] & X \\ s & \in & S := \{x_0(x_1^3 - x_2^2 x_3) + f(x_1, x_2, x_3) = 0\} & & \end{tikzcd}$$

$K^{M,!}_{S,1}$ complex concentrated in hom. degrees 0 and 1

$$K^{M,!}_{S,1}(U) = \Big\{ \underbrace{k(U)^\times}_{1} \to \underbrace{\bigoplus_{u \in U_{(1)}} \mathbb{Z}}_{0} \Big\}, \ \forall U \to S \text{ smooth.}$$

We have an associated Eilenberg–MacLane space $K^{M,!}_{S,1} \in \mathcal{H}(S)$. which induces $M := i_* e^* K^{M,!}_{S,1} \in \mathcal{H}(\underline{X})$ such that $\pi_0^{\mathbb{A}^1}(M)$ is not $\mathbb{A}^1$-invariant.

Aim Construct $\Phi_X(M) \in \mathcal{H}(k)$ such that $\pi_0^{\mathbb{A}^1}(\Phi_X(M))$ is not $\mathbb{A}^1$-invariant.

$$\Phi_X : \mathcal{H}(X) \to \mathrm{PSh}(\mathrm{Sm}_k)$$
$$M \mapsto \Big(U \mapsto \coprod_{s \in \mathrm{Hom}(U,X)} \Gamma(U, s^* M)\Big)$$

In other terms,

$$\begin{tikzcd} \mathcal{H}(X) \arrow[r, "\sim"] & \mathrm{Set}^{\mathrm{cocont}}\Big(\int_{(\mathrm{Sm}_k/X)^{op}} \mathcal{H}(U) \Big/ (\mathrm{Sm}_k/X)^{op}\Big) \arrow[r, "\Gamma"] & \mathrm{PSh}(\mathrm{Sm}_k/X) \arrow[d, "ff_{X,\#}"'] \\ & & \mathrm{PSh}(\mathrm{Sm}_k) \end{tikzcd}$$

where $ff_{X,\#}$ is the left Kan extension along $ff_X : \mathrm{Sm}_k/X \to \mathrm{Sm}_k$.

Rmk $\mathrm{Loc}_{\mathbb{A}^1}(a_{Nis}(\Phi_X(M))) \simeq p_\#(M)$ where $p : X \to \mathrm{Spec}(k)$ and $p_\# \dashv p^* : \mathcal{H}(k) \to \mathcal{H}(X)$.

Def X is $\underline{\mathbb{A}^1\text{-}\overset{\text{rigid}}{\cancel{\text{discrete}}}}$ if $\forall K/k$ field extension, each map $\mathbb{A}^1_K \to X$ is constant.

Prop X is proper and $\mathbb{A}^1$-rigid, $M \in \mathcal{H}(X) \Rightarrow \Phi_X(M) \in \mathcal{H}(k)$.

Proof $\underbrace{\Phi_X(M)(\emptyset) = *}_{\text{because } \mathcal{H}(\emptyset) \text{ is the final category}}$, $\underbrace{\Phi_X(M)(U_1 \sqcup U_2) = \Phi_X(M)(U_1) \times \Phi_X(M)(U_2)}$

because $\mathrm{Hom}(U_1 \sqcup U_2, X) = \mathrm{Hom}(U_1, X) \times \mathrm{Hom}(U_2, X)$
$s \longleftrightarrow (s_1, s_2)$

$\Gamma(U_1 \sqcup U_2, s^* M) = \Gamma(U_1, s_1^* M) \times$
$\times \Gamma(U_2, s_2^* M)$

$$\begin{tikzcd} U' \arrow[r, hook, "j'"] \arrow[d, "e'"'] & V' \arrow[d, "e"] \\ U \arrow[r, hook] & V \end{tikzcd}$$

e : étale Nis. distinguished square

j $\Downarrow$

$$\begin{tikzcd}
\Phi_X(M)(U') & \Phi_X(M)(V') \arrow[l] \\
\Phi_X(M)(U) \arrow[u] \arrow[ur, phantom, "(I)"] & \Phi_X(M)(V) \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"] = \bigsqcup_{V \xrightarrow[t]{} X} \Gamma(V, t^*M)
\end{tikzcd}$$

WLOG V, V' are connected, j, j' have dense images

Lemma $\mathrm{Hom}(V', X) = \mathrm{Hom}(U', X)$, $\mathrm{Hom}(V, X) = \mathrm{Hom}(U, X)$

Proof Rational maps to a proper, $\mathbb{A}^1$-rigid variety are everywhere defined.

Using the lemma, (I) becomes

$$\begin{tikzcd}
\bigsqcup_{U' \xrightarrow{t'} X} \Gamma(U', (t')^*M) & \bigsqcup_{V' \xrightarrow{t'} X} \Gamma(V', (t')^*M) \arrow[l] & \\
 & & \boxed{\bigsqcup_{V \xrightarrow[t]{} X} \Gamma(V', t^*M)} \arrow[ul] \\
\bigsqcup_{U \xrightarrow[t]{} X} \Gamma(U, t^*M) \arrow[uu] \arrow[uur, phantom, "(II)"] & \bigsqcup_{V \xrightarrow[t]{} X} \Gamma(V, t^*M) \arrow[l] \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

e étale $\Rightarrow$ $\boxed{\mathrm{Hom}(V, X) \to \mathrm{Hom}(V', X)}$ is injective

(II) is cartesian iff (III) is Cartesian

$$\begin{tikzcd}
\bigsqcup_{V \to X} \Gamma(U', t^*M) & \bigsqcup_{V \xrightarrow[t]{} X} \Gamma(V', t^*M) \arrow[l] \\
\bigsqcup_{V \to X} \Gamma(U, t^*M) \arrow[u] \arrow[ur, phantom, "(III)"] & \bigsqcup_{V \to X} \Gamma(V, t^*M) \arrow[l] \arrow[u]
\end{tikzcd}$$

Then, (III) is Cartesian iff $\forall\, t : V \to X$,

$$\begin{tikzcd}\Gamma(U', t^*M) & \Gamma(V', t^*M) \arrow[l] \\ \Gamma(U, t^*M) \arrow[u] & \Gamma(V, t^*M) \arrow[l] \arrow[u]\end{tikzcd}$$

is Cartesian, which is true because t^*M is a Nis sheaf.

$\Rightarrow$ $\Phi_X(M)$ is a Nis sheaf.

Similarly, one can show that $\Phi_X(M) \in \mathcal{H}(k)$.

Prop. X proper and $\mathbb{A}^1$-rigid $\Rightarrow$

$$\pi_0^{\mathbb{A}^1}(\Phi_X(M))(U) = \coprod_{U \xrightarrow{j} X} \Gamma(U, \pi_0^{\mathbb{A}^1}(j^*M))$$

Proof $F := \pi_0^{\mathbb{A}^1}(\Phi_X(M)) = \pi_0(\Phi_X(M))$ is the sheafification of $U \mapsto \coprod_{U \xrightarrow{j} X} \pi_0 \Gamma(U, j^*M)$

$G \in PSh(Sm_k)$, $G(U) := \coprod_{U \xrightarrow{j} X} \Gamma(U, \pi_0^{\mathbb{A}^1}(j^*M))$

G is a Nisnevich sheaf (as before), and we have a map $F \to G$. This map is an iso on stalks, because

$$\underbrace{\pi_0 \Gamma(W, j^*M) \xrightarrow{\sim} \Gamma(W, \pi_0(j^*M))} = \Gamma(W, \pi_0^{\mathbb{A}^1}(j^*M))$$

$\forall$ W/k is henselian, local, ess. smooth because $\pi_0(j^*M)$ is the sheafification of $V \mapsto \pi_0 \Gamma(V, j^*M)$

Cor. X proper and $\mathbb{A}^1$-rigid $\Rightarrow$

$\pi_0^{\mathbb{A}^1}(\Phi_X(M))$ is $\mathbb{A}^1$-invariant $\Leftrightarrow$ $\pi_0^{\mathbb{A}^1}(j^*M)$ is $\mathbb{A}^1$-inv. $\forall$ $j: U \to X$

Proof

$$\begin{tikzcd}
\pi_0^{\mathbb{A}^1}(\Phi_X(M))(U) \arrow[r, "(\heartsuit)"] \arrow[d, phantom, "\cong"] & \pi_0^{\mathbb{A}^1}(\Phi_X(M))(U\times\mathbb{A}^1) \arrow[d, phantom, "\cong"] \\
\coprod_{U \xrightarrow{g} X} \Gamma(U, \pi_0^{\mathbb{A}^1}(g^*M)) \arrow[r] & \coprod_{\substack{\{U\times\mathbb{A}^1 \xrightarrow{t} X\} \\ \| \\ \{U \xrightarrow{g} X\}}} \Gamma(U\times\mathbb{A}^1, \pi_0^{\mathbb{A}^1}(t^*M))
\end{tikzcd}$$

$(\heartsuit)$ is an isomorphism iff $\forall\, g: U \to X$,

$$\Gamma(U, \pi_0^{\mathbb{A}^1}(g^*M)) \longrightarrow \Gamma(U\times\mathbb{A}^1, \pi_0^{\mathbb{A}^1}(g^*M))$$

is an isomorphism, i.e. iff $\pi_0^{\mathbb{A}^1}(g^*M)$ is $\mathbb{A}^1$-invariant for each $g: U \to X$.