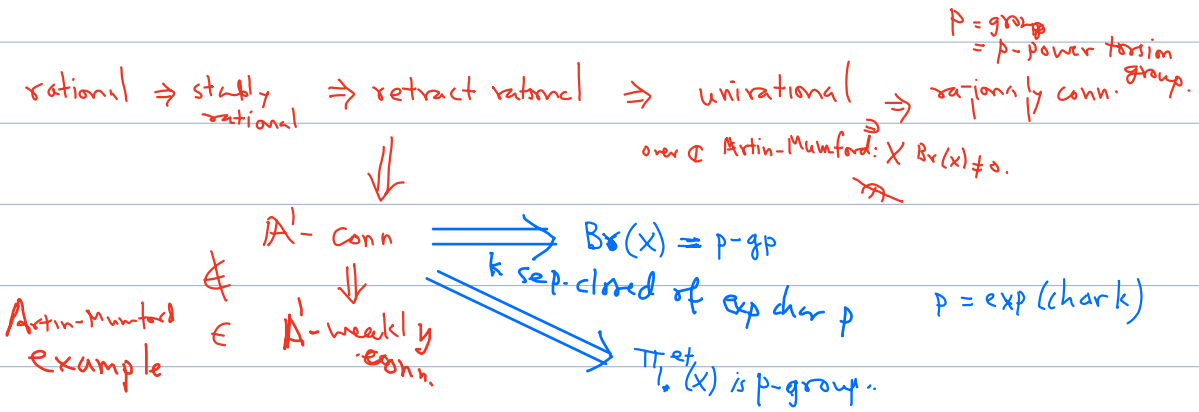


Goals:

Reinterpret h -cobordism in terms of torsors

Compare strong (strict) A' -invariance with A' -locality of classifying spaces (Eilenberg-MacLane spaces)

A' -connectedness \implies cohomological properties

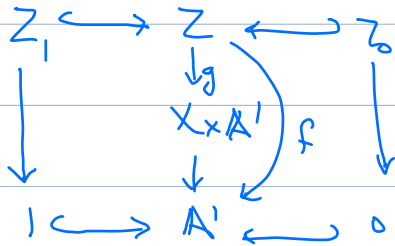


The source of h -cobordisms:

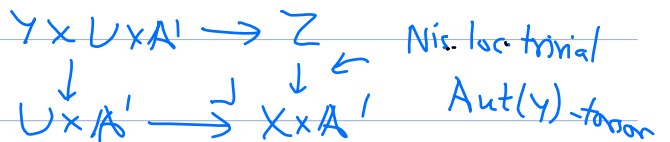
Non A' - h -cobordism constructed earlier: Let $X, Y \in \text{Sm}_k$

-trivial

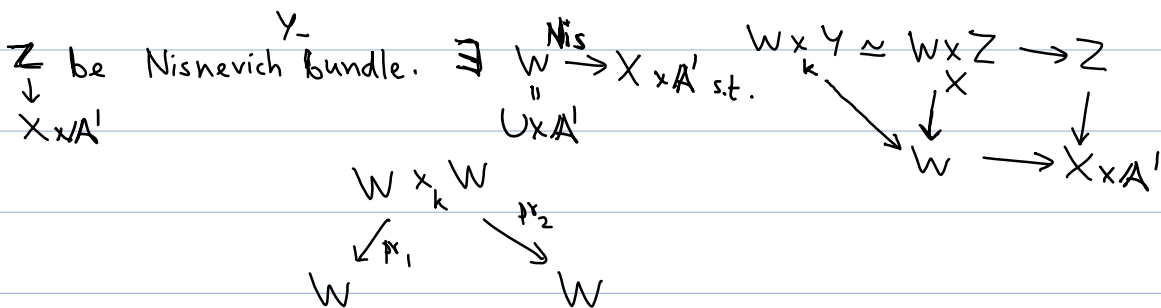
$Z \xrightarrow{g} X \times A'$ is smooth proj morphism.



s.t. $\exists U \xrightarrow{\text{nis}} X$



Simplifying the notation:



$$(W \times_k W) \times_{pr_1, W} (W \times_k Y) \simeq pr_1^*(W \times_k Z) \xrightarrow{\cong} pr_2^*(W \times_k Z) \simeq (W \times_k Y) \times_{W, pr_2} (W \times_k W)$$

$$g \in \text{Aut}(Y) \stackrel{G}{=} \text{Aut}(W \times_k W) = \text{Aut}(Y \times_k (W \times_k W), Y \times_k (W \times_k W))$$

$$G(W) \begin{array}{c} \xrightarrow{pr_1^*} \\ \xleftarrow{pr_2^*} \end{array} G(W \times_k W) \begin{array}{c} \xrightarrow{pr_2^*} \\ \xleftarrow{pr_3^*} \\ \xleftarrow{pr_1^*} \end{array} G(W \times_k W \times_k W)$$

\downarrow
[g]

$$\therefore [g] \in \check{H}_{Nis}^1(X \times A', \text{Aut}(Y))$$

From this perspective a non-trivial h-cobordism as above

$$H_{Nis}^1(X, \text{Aut}(Y)) \xrightarrow{pr_X^*} H_{Nis}^1(X \times A', \text{Aut}(Y))$$

For otherwise: $Z \rightarrow Z'$ $\left[\begin{array}{c} Z \\ \downarrow \\ X \times A' \end{array} \right]$ not in the image.

\downarrow
Cartesian diagram for some $\text{Aut}(Y)$ -torsor $Z' \rightarrow X$

$$\exists h: Z \xrightarrow{h} Z' \times A' \quad \text{Note } h|_{U \times A'}: Z \times (U \times A') \rightarrow (Z' \times A') \times (U \times A')$$

\downarrow \downarrow \downarrow
 $X \times A'$ $X \times A'$ $X \times A'$

\downarrow \downarrow \downarrow
 $Y \times U \times A'$ $Y \times U \times A'$ $Y \times U \times A'$

is iso so (by descent for morphism) h is iso.

Note that $\text{Aut}(Y) \stackrel{G}{=} \text{Aut}(Y)$ is just a sheaf of groups, ^{may be} not ^{represented}

For a sheaf of groups G ,

$$\text{an action on } \mathcal{X}, \quad a: G \times \mathcal{X} \rightarrow \mathcal{X}$$

by smooth \mathbb{k} -group scheme.

categorically free if $G \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is mono of sheaves Δ^{op}

$$(a, x) \longmapsto (a(a, x), x)$$

Let $\mathcal{X} \in \Delta^{\text{op}} \text{Shv}(\text{Sm}_k)$. A G -torsor on \mathcal{X} is $Y \rightarrow \mathcal{X}$

s.t. $a: G \times Y \rightarrow Y$ (left action) is categorically free.

&

$$G \times Y \xrightarrow[a]{\text{pr}_2} Y \rightarrow Y/G \xrightarrow{\sim} \mathcal{X}$$

" G -equivariant
 G -equivariant

Geometrically: if G is k -group scheme, X - k -scheme.

(for $\mathcal{C} = \text{zar, Nis, et}$)

A G -torsor on X \mathcal{C} -locally trivial on X is a tuple $(P \xrightarrow{f} X, G)$ where P is a X -scheme with

scheme theoretically free G -action $a: G \times P \rightarrow P$

(i.e. $G \times P \rightarrow P \times_X P$ is a monomorphism of k -schemes)

$$\begin{array}{ccc} G \times_k W \xrightarrow{\sim} W \times_X P & \rightarrow & P \\ \text{pr}_2 \searrow & \downarrow & \downarrow f \\ & W & \rightarrow X \end{array}$$

faithfully flat G -equivariant map
 \mathcal{C} -cover \Downarrow G -trivial action

We say: P is \mathcal{C} -locally trivial G -torsor on X .

Remarks:

If $\text{Aut}(Y)$ is ^{smooth} k -group scheme, the defs coincide.

so we can use either def.

If Y is proper & $\text{char } k = 0$, $\text{Aut}(Y)$ is k -group scheme.

(need not be k -finite type)

If $k = \text{alg closed}$, Y projective/ k , $\text{Aut}(Y)$ is a smooth k -group scheme, _{loc. of finite type}

Back to h-cobordism: sheaf of groups $\text{Aut}(Y)$ s.t.

$H'_{\text{Nis}}(X, \text{Aut}(Y)) \rightarrow H'_{\text{Nis}}(X \times \mathbb{A}^1, \text{Aut}(Y))$ is not surjective, produces a non-trivial h-cobordism.

This prompts the following question:

Question: Under which conditions on G , p_X^* is a bijection?

$$p_X^*: H'_{\text{Nis}}(X, G) \xrightarrow{\sim} H'_{\text{Nis}}(X \times \mathbb{A}^1, G)$$

Strong \mathbb{A}^1 -invariance and \mathbb{A}^1 -local classifying space:

Def: Let G be a Nisnevich sheaf of groups.

We say G is strongly \mathbb{A}^1 -invariant if $\forall U \in \text{Sm}/k$,

$$p^*: H^i(U, G) \xrightarrow{\sim} H^i(U \times \mathbb{A}^1, G) \quad \text{for } i=0,1.$$

We say G is strongly \mathbb{A}^1 -invariant in étale topology

if $\forall U \in \text{Sm}/k$,

$$p^*: H^i_{\text{ét}}(U, G) \xrightarrow{\sim} H^i_{\text{ét}}(U \times \mathbb{A}^1, G) \quad \text{for } i=0,1.$$

Examples: $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ for a space \mathcal{X} are typical examples of strongly \mathbb{A}^1 -invariant sheaves.

Let G be a Nisnevich sheaf of group.

Let $EG = \check{C}$ ech object associated to $G \rightarrow \text{Spec } k$.

Explicitly $(EG)_n = G \times_k \cdots \times_k G$ $(n+1)$ -times

$$(EG)_{n+1} = G \times \dots \times G \times G \quad (BG)_{n+1} = G^{x_{n+1}}$$

$$(g_0, \dots, g_n) \mapsto (g_0, g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{n-1}^{-1} g_n, g_n)$$

$$(EG)_n = G \times \dots \times G \quad (BG)_n = G^{x_n}$$

$$(g_0, \dots, g_n)$$

$$\vartheta \in G \quad \vartheta(g) = (g_0, \dots, g_{n-1}, g_n \vartheta)$$

$$(EG)_1 \quad (g_0, g_1) \quad G \times G \quad (BG)_1 = G$$

$$(EG)_0$$

$$\downarrow d_0$$

$$\downarrow d_1$$

$$\downarrow d_2$$

$$\downarrow d_3$$

$$\downarrow d_4$$

$$\downarrow d_5$$

$$\downarrow d_6$$

$$\downarrow d_7$$

$$\downarrow d_8$$

$$\downarrow d_9$$

$$\downarrow d_{10}$$

$$\downarrow d_{11}$$

$$\downarrow d_{12}$$

$$(BG)_1 = G$$

$$(BG)_0 = *$$

1) diagonal action $(g_0^{-1} g_1, \dots, g_{n-1}^{-1} g_n, g_n) \xrightarrow{MV} (g_0, g_1, \dots, g_n)$
of G on EG with map to BG

2) action: multiply $n+1$ th co-ordinate & map to BG proj first n -co-ordinates

EG admits extra degeneracy:

$s_i: G \rightarrow G \times G$ (Goerss-Jardine III, Lemma 5.1, EG is simplicially contractible.
- simp. hom. theory

i.e. $EG \rightarrow *$ is w.e.

Clearly $\text{Spec } k \rightarrow EG \rightarrow \text{Spec } k$ is id.

To see: $EG \xrightarrow{f} \text{Spec } k \xrightarrow{s} EG$ is homotopy equivalent to id_{EG}

$$EG \times \Delta[1] \rightarrow EG$$

$$\text{Define } h: (EG)_n \times \Delta[1]_n \rightarrow (EG)_n$$

$$h: \underbrace{G \times \dots \times G}_{n+1 \text{ times}} \times \text{Hom}([n], [1]) \rightarrow \underbrace{G \times \dots \times G}_{n+1 \text{ times}}$$

$$h: (g_0, \dots, g_n) \times \alpha \mapsto (a^{\alpha(0)} g_0, a^{\alpha(1)} g_1, \dots, a^{\alpha(n)} g_n)$$

$$\text{Spec } k \xrightarrow{s} G \xrightarrow{f} \text{Spec } k$$

$$a^0 = \text{id} \quad a^1 = \text{sof}$$

$$h_{n,0} = \text{id} \quad \& \quad h_{n,n+1} = \text{sof}$$

Other identities can be checked which imply that

h is simplicially homotopy from id to sof .

Note $EG \rightarrow BG$ is G -torsor (universal).

BG classifies G -torsors locally trivial in Nisnevich topology.

$$\pi_0(BG(U), *) \simeq H^1(U, G) = \text{set of iso. classes}$$

of G -torsors on U .

Let $BG \rightarrow BG$ be fibrant replacement

$$[U, BG]_{H_s(k)} \simeq H^1(U, G).$$

$$\text{Similarly } [U, BG]_{H_s^{\text{et}}(k)} \simeq H_{\text{et}}^1(U, G)$$

ii

set of iso. classes of étale G -torsors.

Let G be Nisnevich sheaf of groups.

Then BG is A^1 -local iff G is strongly A^1 -invariant.

(simp. fib. replacement of BG)

Similarly if G is étale sheaf of groups:

Then BG is A^1 -local iff G is strongly A^1 -invariant in étale topology.

(simp. fib. replacement of BG)

Notation: let $G = \text{etale sheaf of groups.}$

$$\alpha : (S_{m_k})_{\text{et}} \longrightarrow (S_{m_k})_{\text{Nis}}$$

(adjoint pair) $\alpha^* : \text{Spc}_k \xrightleftharpoons{\alpha_*} \text{Spc}_k^{\text{et}} : \alpha_*$

$$B_{\text{et}} G := \alpha_* (BG^f)$$

where BG^f is etale simplicial fibrant replacement of BG .

Lemma: let G be an etale sheaf of groups. Then

G is strongly A' -invariant in et topology iff $B_{\text{et}} G$ is A' -local.

Thus if G is strongly A' -invariant in et topology then for $U \in S_{m_k}$

$$[\Sigma_s^i \wedge U_+, (B_{\text{et}} G, *)]_{H_{A',*}^{\text{Nis}}(k)} \xrightarrow{\sim} H_{\text{et}}^{i-1}(U, G) \quad i=0,1.$$

Proof: G is strongly A' -inv in etale top. iff BG is A' -local in etale topology.

If BG is A' -local in etale topology, then BG^f is A' -local in etale topology. Now α_* preserves A' -local objects.

To see this:

let \mathcal{X} be A' -local in Spc_k^{et} .

$$[Y, \alpha_* \mathcal{X}]_s \xrightarrow[\text{hence}]{\sim} [Y \times A', \alpha_* \mathcal{X}]_s$$

$$[\alpha^* Y, \mathcal{X}]_{s, \text{et}} \xrightarrow[\text{since } \mathcal{X} \text{ is } A'\text{-local in et tp.}]{\sim} [A'^*(Y \times A'), \mathcal{X}]_{s, \text{et}}$$

$$\xrightarrow[\text{since } \mathcal{X} \text{ is } A'\text{-local in et tp.}]{\sim} [(\alpha^* Y) \times A', \mathcal{X}]_{s, \text{et}}$$

Thus $B_{\text{et}} G := \alpha_*(BG^f)$ is A^1 -local in Spc_k .

Conversely, let $B_{\text{et}} G$ be A^1 -local.

$$\mathcal{X} \in \text{Spc}_k$$

$$\begin{aligned} [(\mathcal{X}, x), (B_{\text{et}} G, *)]_s &\xrightarrow{\simeq} [(\mathcal{X}, x), (B_{\text{et}} G, *)]_{A^1} \\ &\cong (\alpha^*, \alpha_*) \end{aligned}$$

$$[\alpha^*(\mathcal{X}, x), (BG^f, *)]_{s, \text{et}}$$

\therefore for $(\mathcal{X}, x) = \sum_s^i \wedge U_+$ we get

$$[\sum_s^i \wedge U_+, (BG, *)]_{H_{A^1, \text{et}}(k)} \longrightarrow [\sum_s^i \wedge U_+, (B_{\text{et}} G, *)]_{A^1}$$

\cong Prop. 1.16, page 72, MV

$$H_{\text{et}}^{1-i}(U, G)$$

(Morel)

Thm: Let (\mathcal{X}, x) be a pointed space. Then $\pi_i^{A^1}(\mathcal{X}, x)$ is strongly A^1 -invariant for all $i > 0$.

Corollary: Let G be strongly A^1 -invariant in étale topology.

Then G is strongly A^1 -invariant in Nisnevich topology.

Proof: We have $B_{\text{et}} G$ is A^1 -local.

So $\pi_1^s(B_{\text{et}} G, *) \xrightarrow{\simeq} \pi_1^{A^1}(B_{\text{et}} G, *)$ is an iso of sheaves.

For each $U \in \text{Sm}/k$,

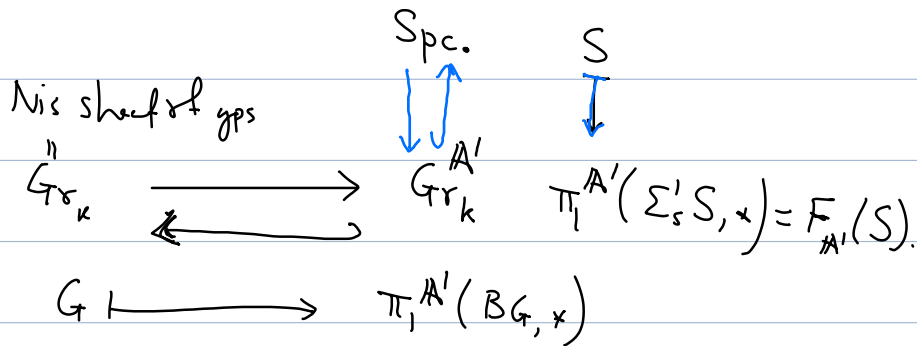
$$[\sum_s^i \wedge U_+, (B_{\text{et}} G, *)]_{A^1} \simeq H_{\text{et}}^0(U, G) = G(U)$$

Thus taking the Nisnevich sheafification of both the presheaves

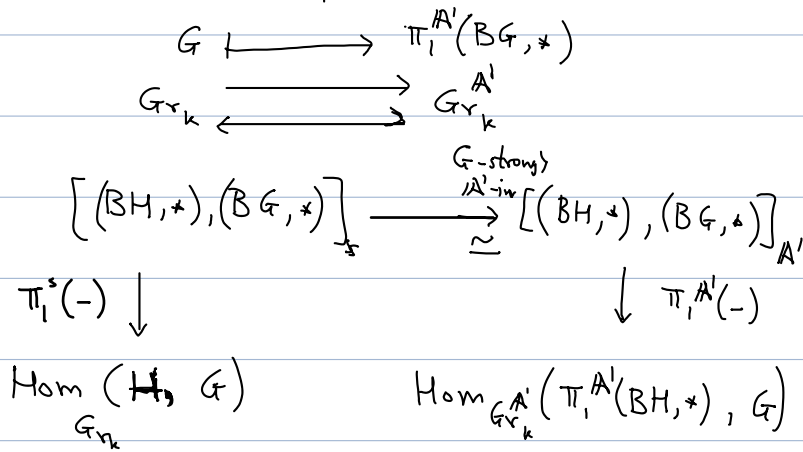
strongly A' -inv. sheaf by Morel $\rightarrow \pi_1^{A'}(B_G, *) \simeq a_{\text{Nis}}(G) = G$

$\therefore G$ is strongly A' -inv in Nisnevich topology.

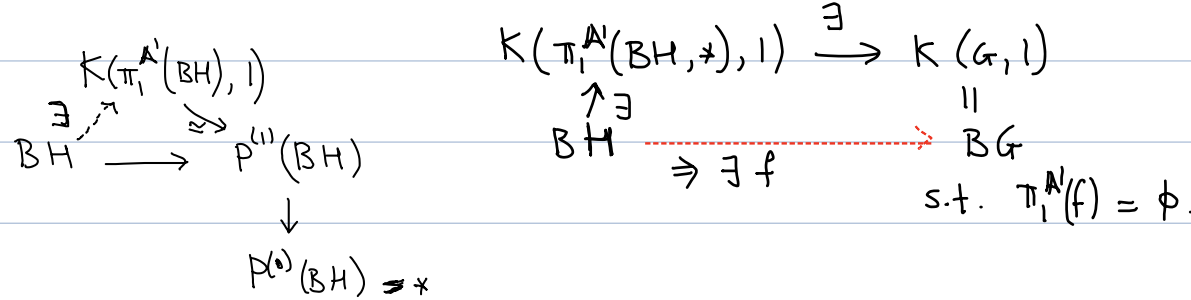
Category of strongly A' -invariant sheaves



Pf: We have adjoint pair



Right vertical map is surjective: Given $\pi_1^{A'}(BH, *) \xrightarrow{\phi} G$



Right vertical map is injective: Let $(BH, *) \xrightarrow[f_*]{f} (BG, *)$ s.t.
 $f_* = g_* : \pi_1^{A'}(BH, *) \rightarrow G$

$$\Rightarrow \begin{array}{c} f \\ = \\ g \end{array} : BH \longrightarrow K(\pi_1^{A'}(BH), 1) \begin{array}{c} \xrightarrow{f_*} \\ \parallel \\ \xrightarrow{g_*} \end{array} K(BG, 1) \xrightarrow{\sim} BG$$

Left vertical map is surj: Given $H \xrightarrow{\phi} G$, let $BH \xrightarrow{B(\phi)} BG$
 then on $\pi_1^S(-)$: $\pi_1^S(BH) \longrightarrow \pi_1^S(BG)$
 $\parallel \quad \quad \quad \parallel$
 $H \xrightarrow{\phi} G$

Inj:

$$BH \xrightarrow{\sim} K(\underbrace{\pi_1^S(BH)}_H, 1) \xrightarrow{\parallel} K(\underbrace{\pi_1^S(BG)}_G, 1) \cong BG$$

Lemma: The category $Gr_k^{A'}$ admits small colimits.

We have adjoint pair $Gr_k \begin{array}{c} \longleftarrow \\ \longrightarrow \end{array} Gr_k^{A'}$

Given a colimit diagram $\{H_i\}$ in $Gr_k^{A'}$,

the $\text{colim}_i H_i$ exists in Gr_k .

Since $\pi_1^{A'}(B(-), *) : Gr_k \rightarrow Gr_k^{A'}$ being left adjoint
 \vee preserves colimits

Hence $\pi_1^{A'}(B(\text{colim}_i H_i)) \cong \text{colim}_i \pi_1^{A'}(BH_i)$.

Def: Given $G_1 \leftarrow H \rightarrow G_2$

diagram of strongly A' -inv sheaves of groups:

$$G_1 \times_H^{A'} G_2 := \pi_1^{A'}(B(G_1 \times_H G_2))$$

"amalgamated sum of G_1 and G_2 over H ." *theng:*

$$G_1 \times_H^{A'} G_2 = \text{colimit in } Gr_k.$$

*Used later
 in "Van-Kampen
 thm" in A' -homotopy
 theory.*

Def: The free strongly A' -inv sheaf of groups on a pointed sheaf of sets (S, s) is

$$F_{A'}(S) = \pi_1^{A'}(\Sigma'_s S).$$

For any strongly A' -invariant sheaf of groups G

$$\text{Hom}_{\text{Gr}_k^{A'}}(F_{A'}(S), G) \xrightarrow{\sim} \text{Hom}_{\text{Sp}_{c,k}}((S, s), G).$$

pf: BG is A' -local \Rightarrow

$$[\Sigma'_s S, BG]_s \xrightarrow{\sim} [\Sigma'_s S, BG]_{A'}$$

$$\pi_1^{A'}(-) \downarrow \cong$$

$$\text{Hom}_{\text{Gr}_k^{A'}}(\pi_1^{A'}(\Sigma'_s S), G)$$

$$[(S, s), \Omega'_s BG]_s$$

$$\downarrow \cong \quad G \simeq \Omega'_s BG_{H_{s,c}(k)}$$

$$\text{Hom}_{\text{Sp}_{c,k}}((S, s), G)$$

$$K(\pi_1^{A'}(\Sigma'_s S), 1) \rightarrow K(G, 1) = BG$$

$$\Sigma'_s \xrightarrow{\quad} P^{(c)}(\Sigma'_s) \xrightarrow{\quad} P'(BG)$$

Proposition: let $X \in \text{Sm}_k$ be \mathbb{A}^1 -connected. and G be an étale sheaf of groups strongly \mathbb{A}^1 -invariant in étale topology.

For two points $x_1, x_2 \in X(k)$, the maps coincide i.e.

$$x_1^* = x_2^* : H_{\text{ét}}^1(X, G) \longrightarrow H_{\text{ét}}^1(\text{Spec } k, G).$$

Let us denote this map by ρ for any rational point.

The natural map

$$\begin{array}{ccc}
 & & H_{\text{ét}}^1(X, G) \xrightarrow{\rho} H_{\text{ét}}^1(\text{Spec } k, G) \\
 & \nearrow & \\
 H_{\text{Nis}}^1(X, G) & \xrightarrow{\text{IU}} & \rho^{-1}(*)
 \end{array}$$

Proof: 1st part: $H_{\text{ét}}^1(X, G) \longrightarrow H_{\text{ét}}^1(\text{Spec } k, G)$

$$\begin{array}{ccc}
 \uparrow & & \uparrow \\
 [X, B_{\text{ét}}G]_{\mathbb{A}^1} & & [\text{Spec } k, B_{\text{ét}}G]_{\mathbb{A}^1} \\
 \downarrow \text{B}_{\text{ét}}G \text{ is } \mathbb{A}^1\text{-fibrant} & & \\
 [X, B_{\text{ét}}G]_s & \dashrightarrow & \\
 \downarrow \tau & &
 \end{array}$$

Choose a representative $\tau : X \rightarrow B_{\text{ét}}G$. The composite map

$$\text{Spec } k \xrightarrow{x_i} X \xrightarrow{\tau} B_{\text{ét}}G$$

factors through $X \rightarrow \pi_0^{\mathbb{A}^1}(X)$.

$$\begin{array}{ccc}
 \parallel & \downarrow & \\
 * = \pi_0^{\mathbb{A}^1}(X) & \xrightarrow{\pi_0^{\mathbb{A}^1}(\tau)} & \pi_0^{\mathbb{A}^1}(B_{\text{ét}}G) \\
 & \text{IU} &
 \end{array}$$

Composing τ with $\text{Spec } k \xrightarrow{x_i} X$ induces the same section $x \rightarrow \pi_0^{\mathbb{A}^1}(B_{\text{ét}}G)$

i.e. an elt in $\pi_0^{\mathbb{A}^1}(B_{\text{ét}}G)(k) = H_{\text{ét}}^1(k, G)$.

$$\tau \longmapsto \pi_0^{\mathbb{A}^1}(\tau)$$

hence independent of $x \in X(k)$.

2nd part: Thinking of G as Nisnevich sheaf, BG is \mathbb{A}^1 -0-connected.

The map $(BG, *) \rightarrow (B_{\text{et}}G, *)$

induces inclusion $* = \pi_0^{\mathbb{A}^1}(BG) \rightarrow \pi_0^{\mathbb{A}^1}(B_{\text{et}}G)$

& iso $\pi_i^{\mathbb{A}^1}(BG) \rightarrow \pi_i^{\mathbb{A}^1}(B_{\text{et}}G)$.

Thus $(BG, *) \rightarrow (B_{\text{et}}G, *)$ is the inclusion of the \mathbb{A}^1 -connected component of the base point.

We have:

$$\begin{array}{ccc}
 [X, BG]_{\mathbb{A}^1} & \xrightarrow{\quad} & [X, B_{\text{et}}G]_{\mathbb{A}^1} \xrightarrow{p} [Spec k, B_{\text{et}}G]_{\mathbb{A}^1} \\
 \left| \begin{array}{l} BG \text{ is } \mathbb{A}^1\text{-local} \\ \hline H'_{Nis}(X, G) \end{array} \right. & & \left| \begin{array}{l} \hline H'_{\text{et}}(X, G) \end{array} \right. \\
 & & \pi_0^{\mathbb{A}^1}(B_{\text{et}}G)(Spec k)
 \end{array}$$

$\left. \begin{array}{l} X \rightarrow B_{\text{et}}G \\ \downarrow \exists \\ \pi_0^{\mathbb{A}^1}(X) \rightarrow \pi_0^{\mathbb{A}^1}(B_{\text{et}}G) \\ \downarrow * \\ \text{Conn. component of } * \end{array} \right\}$

$\bar{p}^{-1}(*) = \{ X \rightarrow B_{\text{et}}G \mid \dots \}$
 \downarrow
 $\exists \text{ as } \pi_0^{\mathbb{A}^1}(BG) = *$

Claim: $[X, BG]_{\mathbb{A}^1} \rightarrow \bar{p}^{-1}(*)$ is bijective.

Clearly 1-1.

Onto: Choose a representative $\tau: X \rightarrow B_{\text{et}}G$ s.t.

$$\begin{array}{ccc}
 X & \xrightarrow{\tau} & B_{\text{et}}G \\
 \downarrow & & \downarrow \\
 \pi_0^{\mathbb{A}^1}(X) & \xrightarrow{*} & \pi_0^{\mathbb{A}^1}(B_{\text{et}}G)
 \end{array}$$

inclusion of connected component of $*$.

X is \mathbb{A}^1 -connected, so has a Postnikov tower

$$\begin{array}{ccc}
 & K(\pi_1^{A^1}(X, x), 1) & \xrightarrow{\exists} K(\pi_1^{A^1}(B_{\text{et}} G, \circ)) \simeq K(G, 1) = BG \\
 \exists \nearrow & \downarrow P^{(1)} & \downarrow \\
 X & X & \rightarrow P^{(1)}(B_{\text{et}} G^{(o)}) \\
 & \downarrow & \\
 & x &
 \end{array}
 \left. \vphantom{\begin{array}{ccc} & K(\pi_1^{A^1}(X, x), 1) & \xrightarrow{\exists} K(\pi_1^{A^1}(B_{\text{et}} G, \circ)) \simeq K(G, 1) = BG \\ \exists \nearrow & \downarrow P^{(1)} & \downarrow \\ X & X & \rightarrow P^{(1)}(B_{\text{et}} G^{(o)}) \\ & \downarrow & \\ & x & \end{array}} \right\} \begin{array}{l} \text{functoriality} \\ \text{of Postnikov} \\ \text{tower} \end{array}$$

Proposition: Let k be a separably closed field of exp. char p .

If X is A^1 -connected, then X admits no non-trivial finite étale Galois covers of degree coprime to p .

Hence $\pi_1^{\text{et}}(X, x)$ is pro- p -group.

Pf: Let $Y \rightarrow X$ be a finite étale Galois cover associated to a group G of order coprime to p .

Since G is an étale group scheme of order coprime to p .

By [MV, §4, Prop 3.1, page 137], $B_{\text{et}} G$ is A^1 -local.

Thus G is strongly A^1 -invariant in étale topology.

Hence by above proposition

$$H_{\text{Nis}}^1(X, G) \xrightarrow{\simeq} H_{\text{et}}^1(X, G) \rightarrow H_{\text{et}}^1(\text{Spec } k, G)$$

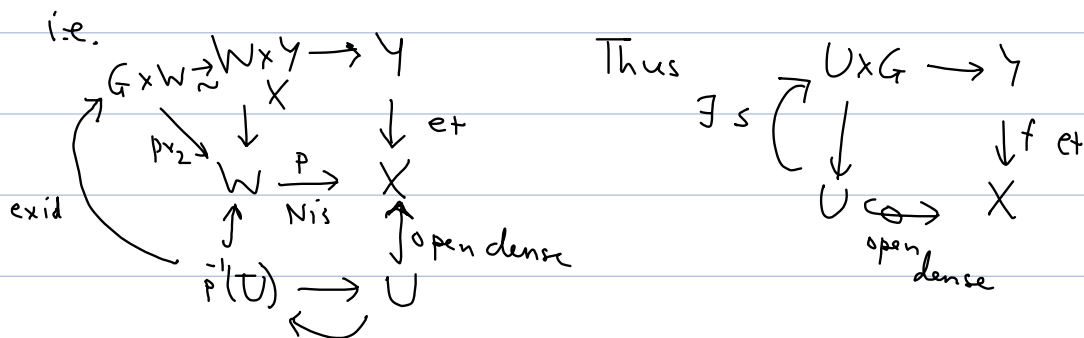
$\begin{array}{c} \uparrow \\ k \text{ sep. closed} \\ * \end{array}$

$\therefore Y \xrightarrow{f} X$ is Nisnevich locally trivial G -torsor.

Claim: $Y \xrightarrow{f} X$ admits a section i.e. $\exists s: X \rightarrow Y$ s.t. $f \circ s = \text{id}$.

WLOG we can assume that X is irreducible smooth variety / k .

By assumption that $Y \rightarrow X$ is Nis. locally trivial

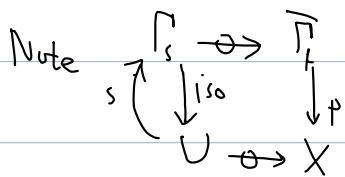
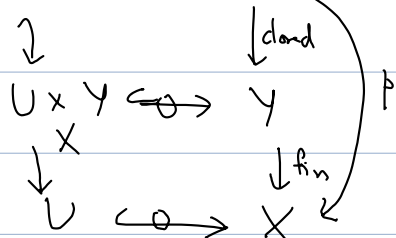


Consider

$\Gamma_s \hookrightarrow U \times_X Y$ the graph of s

& $\overline{\Gamma}_s$ the closure of Γ_s in Y .

$\Gamma_s \hookrightarrow \overline{\Gamma}_s =$ reduced closed subscheme



Then $p: \overline{\Gamma}_s \rightarrow X$ is such that $p|_{\Gamma_s}: \Gamma_s \rightarrow U$ is iso.

$\Rightarrow p$ is birational.

Further p is finite, hence proper.

Thus $p: \overline{\Gamma}_S \rightarrow X$ is a birational, proper map of integral schemes
 \Rightarrow each fiber of p is connected.

but p is finite, hence quasi-finite.
 \Rightarrow each fiber is singleton.

Also p is proper, hence closed, & $p(\overline{\Gamma}_S) \supset U$ dense in X :
 $\therefore p(\overline{\Gamma}_S) = X$
 i.e. p is surjective.

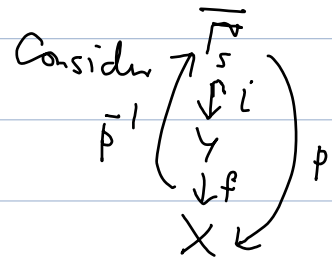
Thus $p: \overline{\Gamma}_S \rightarrow X$ is proper birational

bijective map

$\Rightarrow p$ is iso.

A. Mathew's

Zariski's main Cor. 3.9
 thm application



Thus $\begin{matrix} \text{is } \overline{p}^{-1} \\ \text{is } \overline{p}^{-1} \\ \text{is } \overline{p}^{-1} \end{matrix} : X \rightarrow \overline{\Gamma}_S \rightarrow Y$ s.t. $f \circ g = \text{id}_X$

Hence $\pi_1^{\text{et}}(X, x)$ is pro-p-group.

Strict A' -invariance and A' -local Eilenberg-MacLane spaces

Def: Let A be a Nisnevich sheaf of abelian groups.

A is strictly A' -invariant if $\forall U \in \text{Sm}_k$

$$H_{\text{Nis}}^i(U, A) \xrightarrow{\cong} H_{\text{Nis}}^i(U \times A', A) \quad \forall i \geq 0.$$

Given étale sheaf A , A is strictly A' -inv in étale topology if

$$H_{\text{et}}^i(U, A) \xrightarrow{\cong} H_{\text{et}}^i(U \times A', A) \quad \forall i \geq 0.$$

A is strictly A' -inv $\Leftrightarrow K(A, i)$ is A' -local $\forall i \geq 0$

A is strictly A' -inv in et $\Leftrightarrow K(A, i)$ is A' -local $\forall i \geq 0$
in Spc_k^{et} .

Def: Let A be étale sheaf $K(A, i) \in \text{Spc}_k^{\text{et}}$.

Let $K(A, i)^f =$ fibrant replacement of $K(A, i)$ in Spc_k^{et} .

Define $K_{\text{et}}(A, i) = \alpha_* K(A, i)^f \in \text{Spc}_k$.

Lemma: Let A be étale sheaf of abelian groups. Then

A is strictly A' -inv in étale topology iff $K_{\text{et}}(A, i)$ is A' -local $\forall i \geq 0$.

Moreover if A is strictly A' -inv étale top, then $\forall U \in \text{Sm}_k$

$$\left[\sum_s^j \wedge U_s, K_{\text{et}}(A, i) \right]_{A'} \xrightarrow{\cong} H_{\text{et}}^{i-j}(U, A)$$

$\forall 0 \leq j \leq i$.

Pf: Similar as before:

$$\begin{array}{ccc}
 \left[\sum_s^j \wedge U_+, K_{\text{et}}(A, i) \right] & \simeq & \left[\alpha^* \left(\sum_s^j \wedge U \right), K(A, i)^f \right] \\
 & & \text{\scriptsize } A' \text{ adjunction} \quad \Big| \quad \text{\scriptsize } H_{\mathbb{A}^1}^{\text{et}}(U) \\
 & \searrow \simeq & \left[\sum_s^j \wedge U_+, K(A, i)^f \right] \\
 & & \text{\scriptsize } H_{\mathbb{A}^1}^{\text{et}}(k) \\
 & & \text{\scriptsize } \mathbb{R} \text{ } K(A, i) \text{ is } A' \text{-local} \\
 & & \left[\sum_s^j \wedge U_+, K(A, i) \right] \\
 & & \text{\scriptsize } H_S^{\text{et}}(k) \\
 & & \Big| \quad \mathbb{R} \\
 & & H_{\text{et}}^{i-j}(U, A)
 \end{array}$$

book: Thm 5.46

Thm (Morel): Let A be strongly A' -invariant sheaf of abelian groups.

Then A is strictly A' -invariant.

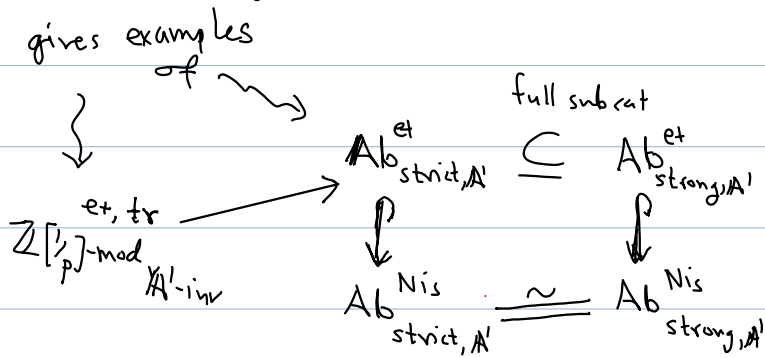
Pf: (Sketch)

$$\begin{array}{l}
 H^*(U, A) \simeq H^*(C_{R_S}(U; A)) \\
 C_{R_S}(U; A) \longrightarrow C_{R_S}(U \times A'; A) \\
 \text{\scriptsize quasi-isomorphism}
 \end{array}
 \left. \vphantom{\begin{array}{l} H^*(U, A) \simeq H^*(C_{R_S}(U; A)) \\ C_{R_S}(U; A) \longrightarrow C_{R_S}(U \times A'; A) \end{array}} \right\} \forall U \in \mathcal{S}_{m, k}$$

Cor: Let A be an étale sheaf of abelian groups strongly A' -invariant in ét. top. Then the underlying Nisnevich sheaf A is strictly A' -invariant.

Pf: strongly A' -inv in ét top \Rightarrow strongly A' -inv. in Nis-top
 \Downarrow Thm
 strict A' -inv.

Summarizing :



Lemma: let k be a field so that $\text{char } k = p$. Let A be etale sheaf $\mathbb{Z}[\frac{1}{p}]$ -modules that is \mathbb{A}' -invariant. Then A is strictly \mathbb{A}' -invariant for etale topology.

Pf: Given a s.e.s. of etale sheaves of $\mathbb{Z}[\frac{1}{p}]$ -modules

$$0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$$

if 2 of them are strictly \mathbb{A}' -inv in et top then so is 3rd.

Consider $0 \rightarrow A_{\text{tors}} \rightarrow A \rightarrow A \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow A \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \rightarrow 0$

Note A_{tors} as torsion coprime to p .

Reduce to show lemma for A either

- 1) A is a sheaf of \mathbb{Q} -vector spaces
 - 2) A is torsion coprime to p .
- } etale sheaves with transfers.

In case 1) $\forall i: H_{\text{et}}^i(U, A) \cong H_{\text{Nis}}^i(U, A)$. & this is \mathbb{A}' -inv. Ref: Prop 14.23 lectures on motivic coh.

2) [MVW, Thm 7.20]: Suslin's rigidity thm: Thm. 13.8

A is locally constant: $\pi^* \pi_* A \xrightarrow{\sim} A$ is iso where

$$\pi^*: \text{Sh}_{\text{et}}(\text{Et}/k) \xrightarrow{\sim} \text{Sh}_{\text{et}}(\text{Sm}/k): \pi_*$$

Then by [SGA4, Tome 3, XV, cor 2.2]

$H_{\text{et}}^i(-, A)$ is A' -invariant.

Prop: Let k be a field of exp char p . Let $X \in \text{Sm}/k$.

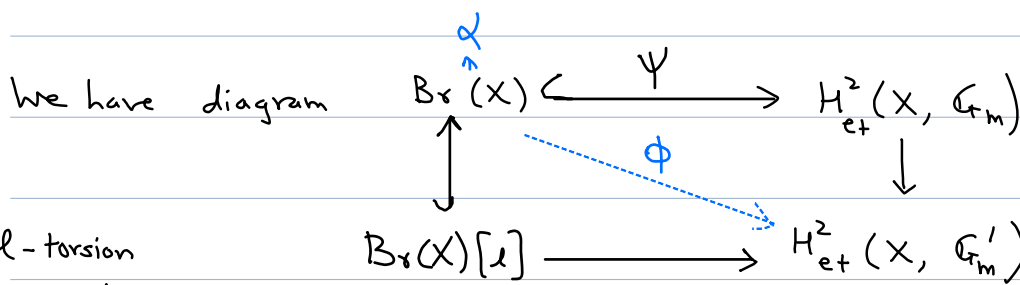
Let $l \neq p$, $G'_m = G_m \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$. Then $\text{Br}(X)[l] \hookrightarrow H_{\text{et}}^2(X, G'_m)$.

Proof: Let $G'_m =$ etale sheaf associated to the presheaf
actually a sheaf

$$U \mapsto \mathcal{O}(U)^\times \otimes \mathbb{Z}[\frac{1}{p}]$$

char $k=0$: $G'_m = G_m$

We know $\text{Br}(X) \hookrightarrow H_{\text{et}}^2(X, G_m)$



Claim: for l coprime to p : ϕ is injective on $\text{Br}(X)[l]$.

For if $\alpha \in \text{Br}(X)[l]$ s.t. $0 = \phi(\alpha)$.

We have $l^n \cdot \alpha = 0$ in $\text{Br}(X)$ for some $n \geq 1$.

G'_m is $\mathbb{Z}[\frac{1}{p}]$ -modules $\Rightarrow H_{\text{et}}^2(X, G'_m)$ is also $\mathbb{Z}[\frac{1}{p}]$ -modules.

So $\phi(\alpha) = 0 \Rightarrow \exists m$ s.t. $p^m \Psi(\alpha) = 0$ in $H_{\text{et}}^2(X, G_m)$.

$\Psi(p^m \alpha) = 0 \Rightarrow p^m \alpha = 0$ in $\text{Br}(X)$.

But $\ell^m \cdot \alpha = 0$, since α in $\text{Br}(X)[\ell]$.

$\therefore \alpha = 0$ in $\text{Br}(X)$

i.e. $\phi: \text{Br}(X)[\ell] \rightarrow H_{\text{et}}^2(X, \mathbb{G}_m')$ is injective for $\ell \neq p$.

Prop. 4-3.8: k field ex p char $k = p$. $x \in X(k)$. X \mathbb{A}^1 -connected.

Then $X \xrightarrow[\pi]{\pi} \text{Spec } k$ induces isomorphism

$$H_{\text{et}}^2(X, \mathbb{G}_m') \cong H_{\text{et}}^2(k, \mathbb{G}_m')$$

In particular if k is separably closed, $\text{Br}(X)$ is p -group.

pf:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \text{---} & \text{---} & \text{---} & \\ & \text{---} & \text{---} & \text{---} & \\ H_{\text{et}}^2(k, \mathbb{G}_m') & \xrightarrow{\cong} & H^2(X, \mathbb{G}_m') & \xrightarrow{\cong} & H_{\text{et}}^2(k, \mathbb{G}_m') \\ \parallel & & \parallel & & \parallel \\ [\mathbb{S}^1 \wedge \text{Spec } k, K_{\text{et}}(\mathbb{G}_m', 2)] & \rightarrow & [X, K_{\text{et}}(\mathbb{G}_m', 2)] & \rightarrow & [\text{Spec } k, K_{\text{et}}(\mathbb{G}_m', 2)] \end{array}$$

To show 1st map is onto:

$$\text{Let } X \xrightarrow{\tau} K_{\text{et}}(\mathbb{G}_m', 2)$$

$$\downarrow \quad \downarrow \\ * = \pi_0^{\mathbb{A}^1}(X) \xrightarrow{\pi_0^{\mathbb{A}^1}(\tau)} \pi_0^{\mathbb{A}^1}(K_{\text{et}}(\mathbb{G}_m', 2))$$

$$\therefore \pi_0^{\mathbb{A}^1}(\tau) \in [\text{Spec } k, K_{\text{et}}(\mathbb{G}_m', 2)]$$

$$\text{s.t. } \pi_0^{\mathbb{A}^1}(\tau) \mapsto \tau \mapsto \pi_0^{\mathbb{A}^1}(\tau)$$

Thus $H_{\text{et}}^2(X, \mathbb{G}_m') \rightarrow H_{\text{et}}^2(k, \mathbb{G}_m')$ is bijective.

$$\text{Br}(X) \longleftrightarrow H_{\text{et}}^2(X, \mathbb{G}_m)$$

↑

↓

$$\text{for } l \neq p: \text{Br}(X)[l] \longleftrightarrow H_{\text{et}}^2(X, \mathbb{G}_m^l) \simeq H_{\text{et}}^2(k, \mathbb{G}_m^l)$$

|| k sep. closed.
0

$$\therefore \text{Br}(X)[l] = 0 \text{ for } l \neq p.$$

$$\therefore \text{Br}(X) \text{ is } p\text{-torsion.}$$

Example: Let $X = K3$ surface / k . $\text{Br}(X) \neq 0$. $\Rightarrow X$ is not \mathbb{A}^1 -conn.

Let $k = \text{alg closed field of char exponential } p$

Let X smooth proper / k .

Let $k = \mathbb{C}$:

$$0 \rightarrow \left(\frac{H^2(X, \mathbb{Z})}{\text{Pic}(X)} \right) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H^2(X, \mathbb{G}_m) \rightarrow H^3(X, \mathbb{Z})_{\text{tors}} \rightarrow 0$$

is exact.

(this follows from $0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\text{exp}} \mathcal{O}_X^\times \rightarrow 0$) & then $\text{Tor}_1(-, \mathbb{Q}/\mathbb{Z})$

Thus $H^2(X, \mathbb{G}_m) \simeq \underbrace{(\mathbb{Q}/\mathbb{Z})^{b_2-p}}_{\text{non-canonically tors.}} \oplus H^3(X, \mathbb{Z})_{\text{tors.}}$ \leftarrow note $\text{Tor}_1(M, \mathbb{Q}/\mathbb{Z}) \cong M_{\text{tors}}$

So if $b_2 > p$ (e.g. $X = K3$ surface $b_2 = 22$ & $p = 20$)

X is not \mathbb{A}^1 -connected.

for char $k > 0$, $l \neq p$

$$H_{\text{et}}^2(X, \mathbb{G}_m)[l] \simeq \left(\frac{\mathbb{Q}}{\mathbb{Z}} \right)^{b_2-p} \oplus M$$

where $M = \text{finite } l\text{-group}$

Example: Let G = simply connected, semi-simple alg. group over a field k , $\text{exp char } k = p$.

$$G^+(k) = \langle f(G_a(k)) \mid f: G_a \rightarrow G \text{ homomorphism} \rangle_{\text{gen.}}$$

$$G(k) \rightarrow \frac{G(k)}{G^+(k)} = W(k, G) = \text{"Whitehead group!"}$$

Kneser-Tits problem: For which groups G , $W(k, G) = 1$.

We say a group G is W -trivial iff $W(L, G) = 1 \quad \forall L/k$.

Question (Gille) Can we characterize G s.t. G is W -trivial?

Note: G is W -trivial $\Rightarrow A^1$ -chain connected $\Rightarrow A^1$ -connected.

(Gille: in addition to above assumptions if G is split, then G is W -trivial \Leftrightarrow hence A^1 -conn.)

By what we saw today, G - W -trivial $\Rightarrow \text{Br}(G)$ is p -torsion.

END!