Goals:

Reinterpret h-cobordism in terms of torsors

Compare strong (strict) $\mathbb{A}^1$-invariance with $\mathbb{A}^1$-locality of classifying spaces (Eilenberg-MacLane spaces)

$\mathbb{A}^1$-connectedness $\Longrightarrow$ cohomological properties

p = group = p-power torsion group.

rational $\Rightarrow$ stably rational $\Rightarrow$ retract rational $\Rightarrow$ unirational $\Rightarrow$ rationally conn.

over $\mathbb{C}$ Artin-Mumford: $\exists\, X$ $Br(x) \neq 0$.

$\Downarrow$

$\mathbb{A}^1$-conn $\Longrightarrow$ $Br(X) = p$-gp

k sep. closed of exp char p $\qquad p = \exp(\operatorname{char} k)$

$\Downarrow$

Artin-Mumford example $\notin$ / $\in$ $\mathbb{A}^1$-weakly conn. $\Longrightarrow$ $\pi_1^{et}(X)$ is p-group.

The source of h-cobordisms:

Non-trivial $\mathbb{A}^1$-h-cobordism constructed earlier: Let $X, Y \in Sm_k$

$Z \xrightarrow{g} X \times \mathbb{A}^1$ is smooth proj morphism.

$$\begin{tikzcd}
Z_1 \arrow[r, hook] \arrow[dd] & Z \arrow[d, "g"] \arrow[dd, bend left, "f"] & Z_0 \arrow[l, hook'] \arrow[dd] \\
 & X \times \mathbb{A}^1 \arrow[d] & \\
1 \arrow[r, hook] & \mathbb{A}^1 & 0 \arrow[l, hook']
\end{tikzcd}$$

s.t. $\exists\ U \xrightarrow{nis} X$

$$\begin{tikzcd}
Y \times U \times \mathbb{A}^1 \arrow[r] \arrow[d] & Z \arrow[d] \\
U \times \mathbb{A}^1 \arrow[r] & X \times \mathbb{A}^1
\end{tikzcd}$$

Nis. loc trivial $Aut(Y)$-torsor

Simplifying the notation:

$$\begin{tikzcd}
Z \arrow[d] \\
X \times \mathbb{A}^1
\end{tikzcd}$$

Y – be Nisnevich bundle. $\exists$ $W \xrightarrow{Nis} X \times \mathbb{A}^1$ s.t. ($W = U \times \mathbb{A}^1$)

$$\begin{tikzcd}
W \times_k Y \simeq W \times_X Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\times"] & Z \arrow[d] \\
W \arrow[r] & X \times \mathbb{A}^1
\end{tikzcd}$$

$$\begin{tikzcd}
 & W \times_k W \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
W & & W
\end{tikzcd}$$

$$(W \times_k W) \times_{pr_1, W} (W \times_k Y) \simeq pr_1^*(W \times_X Z) \xrightarrow[\sim]{g} pr_2^*(W \times_X Z) \simeq (W \times_k Y) \times_{W, pr_2} (W \times_k W)$$

$$g \in \underset{\overset{\parallel}{G}}{\mathrm{Aut}(Y)}(W \times_k W) = \mathrm{Aut}(Y \times_k (W \times_k W), Y \times_k (W \times_k W))$$

$$G(W) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \underset{\underset{[g]}{\Psi}}{G(W \times_k W)} \overset{p_{12}^*,\ p_{23}^*,\ p_{13}^*}{\rightrightarrows} G(W \times_k W \times_k W)$$

$\therefore \quad [g] \in \check{H}^1_{Nis}(X \times \mathbb{A}^1, \mathrm{Aut}(Y))$.

From this perspective a non-trivial h-cobordism as above

$$H^1_{Nis}(X, \mathrm{Aut}(Y)) \xrightarrow{p_X^*} H^1_{Nis}(X \times \mathbb{A}^1, \mathrm{Aut}(Y)) \ni \left[\begin{matrix} Z \\ \downarrow \\ X \times \mathbb{A}^1 \end{matrix}\right] \text{ not in the image.}$$

For otherwise:
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \times \mathbb{A}^1 \arrow[r] & X \end{tikzcd}$$
cartesian diagram for some $\mathrm{Aut}(Y)$-torsor $\begin{matrix} Z' \\ \downarrow \\ X \end{matrix}$

$$\exists \quad \begin{tikzcd} Z \arrow[rr, "\exists h"] \arrow[dr] & & Z' \times \mathbb{A}^1 \arrow[dl] \\ & X \times \mathbb{A}^1 & \end{tikzcd}$$

Note $h|_{U \times \mathbb{A}^1} : \underset{\underset{Y \times U \times \mathbb{A}^1}{\overset{\wr\wr}{}}}{Z \times_{X \times \mathbb{A}^1} (U \times \mathbb{A}^1)} \to \underset{\underset{Y \times U \times \mathbb{A}^1}{\overset{\wr\wr}{}}}{(Z' \times \mathbb{A}^1) \times_{X \times \mathbb{A}^1} (U \times \mathbb{A}^1)}$

is iso so (by descent for morphism) h is iso.

Note that $\underset{\overset{\parallel}{G}}{\mathrm{Aut}(Y)}$ is just a sheaf of groups, may not be represented by smooth k-group scheme.

For a sheaf of groups G,

an action on $\mathcal{X}$, $\quad a : G \times \mathcal{X} \longrightarrow \mathcal{X}$

categorically free if $\quad G \times \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is mono of Δ^{op}-sheaves

$$(\alpha, x) \longmapsto (a(\alpha, x), x)$$

Let $\mathcal{X} \in \Delta^{op} Shv(Sm/_k)$. A G-torsor on $\mathcal{X}$ is $\mathcal{Y} \to \mathcal{X}$

s.t. $a: G \times \mathcal{Y} \to \mathcal{Y}$ (Left action) is categorically free.

&

$$G \times \mathcal{Y} \overset{pr_2}{\underset{a}{\rightrightarrows}} \mathcal{Y} \to \underbrace{\mathcal{Y}/_G}_{\text{Co-equilizer}} \xrightarrow{\sim} \mathcal{X}.$$

Geometrically: if G is k-group scheme, X - k-scheme.

(for $\tau =$ Zar, Nis, et)

A G-torsor on X τ-locally trivial on X is a tuple $(P \xrightarrow{f} X, G)$ where P is a X-scheme with

<u>scheme theoretically free</u> G-action $a: G \times P \to P$

(i.e. $G \times P \to P \times_X P$ is a monomorphism of k-schemes)

$$\begin{tikzcd}
G \times_k W \arrow[r, "\sim"] \arrow[dr, "pr_2"'] & W \times_X P \arrow[r] \arrow[d] & P \arrow[d, "f"] \\
& W \arrow[r, "\tau\text{-cover}"'] & X
\end{tikzcd}$$

$\exists$ $W \to X$ (τ-cover) s.t.

faithfully flat G-equivariant map

$\circlearrowright$ G-trivial action

We say: P is τ-locally trivial G-torsor on X.

Remarks:

If $\mathrm{Aut}(Y)$ is smooth k-group scheme, the defs coincide.

so we can use either def.

If Y is proper & char $k = 0$, $\mathrm{Aut}(Y)$ is k-group scheme.

(need not be k-finite type)

If k = alg closed, Y projective$/k$, $\mathrm{Aut}(Y)$ is a smooth k-group scheme, loc. of finite type

Back to h-cobordism: sheaf of groups $Aut(Y)$ s.t.

$H^1_{Nis}(X, Aut(Y)) \to H^1_{Nis}(X \times \mathbb{A}^1, Aut(Y))$ is not surjective, produces a non-trivial h-cobordism.

This prompts the following question:

<u>Question:</u> Under which conditions on G, p_X^* is a bijection?

$$p_X^* : H^1_{Nis}(X, G) \xrightarrow{\sim} H^1_{Nis}(X \times \mathbb{A}^1, G)$$

Strong $\mathbb{A}^1$-invariance and $\mathbb{A}^1$-local classifying space:

<u>Def:</u> Let G be a Nisnevich sheaf of groups.

We say G is strongly $\mathbb{A}^1$-invariant if $\forall\, U \in Sm/k$,

$$p^* : H^i(U, G) \xrightarrow{\simeq} H^i(U \times \mathbb{A}^1, G) \quad \text{for } i = 0, 1.$$

We say G is strongly $\mathbb{A}^1$-invariant in etale topology if $\forall\, U \in Sm/k$,

$$p^* : H^i_{et}(U, G) \xrightarrow{\simeq} H^i_{et}(U \times \mathbb{A}^1, G) \quad \text{for } i = 0, 1.$$

<u>Examples:</u> $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ for a space $\mathcal{X}$ are typical examples of strongly $\mathbb{A}^1$-invariant sheaves.

Let G be a Nisnevich sheaf of group.

Let EG = Čech object associated to $G \longrightarrow \operatorname{Spec} k$.

Explicitly $(EG)_n = G \times_k \cdots \times_k G$ $(n+1)$-times

$(EG)_{n+1} = G \times \cdots\cdots \times G \times G \qquad (BG)_{n+1} = G^{\times n+1}$

$(g_0, \ldots g_n) \longmapsto (g_1 g_1^{-1}, g_1 g_2^{-1}, \ldots, g_{n-1} g_n^{-1}, g_n)$

$(EG)_n = G \times \cdots\cdots \times G \circlearrowleft G \qquad (BG)_n = G^{\times n}$

$(g_0, \ldots\ldots, g_n)$

$g \in G \quad g(\bar{g}) = (g_0, \ldots, g_{n-1}, g_n \cdot g)$

$(EG)_1 \qquad G \times G \qquad (BG)_1 = G$

$(EG)_0 \qquad (g_0, g_1) \mapsto_{d_0} g_0 \qquad pr_1 = d_0,\ pr_2 = d_1 : G \times G \to G, \quad s_0 = \Delta \qquad (g_0, g_1) \mapsto_{d_1} g_1 \qquad g \mapsto_{s_0} (g, g) \qquad (BG)_0 = *$

1) diagonal action of G on EG: $(g_1, \ldots, g_n) g = (g_0 g, \ldots, g_n \cdot g) \overset{MV}{\longmapsto} (g_0 \cdot g_1^{-1}, \ldots, g_{n-1} g_n^{-1}, g_n \cdot g)$.
with map to BG

2) action: multiply $n+1$ th co-ordinate & map to BG proj first n-co-ordinates

EG admits extra degeneracy:
$s_{-1} : G \xrightarrow{\Delta} G \times G$ (Goerss-Jardine – simp. homo theory III, Lemma 5.1), EG is simplicially contractible.

i.e. $EG \to *$ is w.e.

Clearly $\operatorname{Spec} k \longrightarrow EG \longrightarrow \operatorname{Spec} k$ is id.

To see: $EG \xrightarrow{f} \operatorname{Spec} k \xrightarrow{s} EG$ is homotopy equivalent to id_{EG}

$EG \times \Delta[1] \longrightarrow EG$

Define $h : (EG)_n \times \Delta[1]_n \longrightarrow (EG)_n$

$h : \underbrace{G \times \cdots \times G}_{n+1 \text{ times}} \times \operatorname{Hom}([n], [1]) \longrightarrow \underbrace{G \times \cdots \times G}_{n+1 \text{ times}}$

$h : (g_0, \ldots, g_n) \times \alpha \longmapsto (a^{\alpha(0)} g_0, a^{\alpha(1)} g_1, \ldots, a^{\alpha(n)} g_n)$

$\operatorname{Spec} k \xrightarrow{s} G \xrightarrow{f} \operatorname{Spec} k$

$a^0 = \mathrm{id} \qquad a^1 = s \circ f$

$h_{n,0} = \mathrm{id} \quad \& \quad h_{n,n+1} = s \circ f$

Other identities can be checked which imply that h is simplicially homotopy from id to $s \circ f$.

Note $EG \longrightarrow BG$ is G-torsor (universal).

BG classifies G-torsors locally trivial in Nisnevich topology.

$\pi_0(\mathcal{B}G(U), *) \simeq H^1(U, G)$ = set of iso. classes of G-torsors on U.

Let $BG \longrightarrow \mathcal{B}G$ be fibrant replacement

$$[U, \mathcal{B}G]_{H_s(k)} \simeq H^1(U, G).$$

Similarly $[U, \mathcal{B}G]_{H_s^{et}(k)} \simeq H^1_{et}(U, G)$

$\quad\quad\quad\quad ||$

set of iso. classes of etale G-torsors.

Let G be Nisnevich sheaf of groups.

Then $\mathcal{B}G$ is $\mathbb{A}^1$-local iff G is strongly $\mathbb{A}^1$-invariant.

$||$

(simp. fib. replacement of BG)

Similarly if G is etale sheaf of groups:

Then $\mathcal{B}G$ is etale $\mathbb{A}^1$-local iff G is strongly $\mathbb{A}^1$-invariant in etale topology.

$||$

(simp. fib. replacement of BG)

Notation: Let G = etale sheaf of groups.

$$\alpha : (Sm_k)_{et} \longrightarrow (Sm_k)_{Nis}$$

(adjoint pair) $\alpha^* : Spc_k \rightleftarrows Spc_k^{et} : \alpha_*$

$$B_{et} G := \alpha_* (BG^f)$$

where BG^f is etale simplicial fibrant replacement of BG.

<u>Lemma:</u> Let G be an etale sheaf of groups. Then G is strongly $\mathbb{A}^1$-invariant in et topology iff $B_{et}G$ is $\mathbb{A}^1$-local.

Thus if G is strongly $\mathbb{A}^1$-invariant in et topology then for $U \in Sm_k$

$$[\Sigma_s^i \wedge U_+, (B_{et}G, *)]_{H^{Nis}_{\mathbb{A}^1,*}(k)} \xrightarrow{\sim} H^{1-i}_{et}(U, G) \qquad i = 0, 1.$$

<u>Proof:</u> G is strongly $\mathbb{A}^1$-inv in etale top. iff BG is $\mathbb{A}^1$-local in etale topology.

If BG is $\mathbb{A}^1$-local in etale topology, then BG^f is $\mathbb{A}^1$-local is etale topology. Now α_* preserves $\mathbb{A}^1$-local objects.

To see this:

Let $\mathcal{X}$ be $\mathbb{A}^1$-local in Spc_k^{et}.

$$\begin{array}{ccc}
[\mathcal{Y}, \alpha_* \mathcal{X}]_s & \xrightarrow[\text{hence}]{\simeq} & [\mathcal{Y} \times \mathbb{A}^1, \alpha_* \mathcal{X}]_s \\
\wr\wr & & \wr\wr \\
[\alpha^* \mathcal{Y}, \mathcal{X}]_{s,et} & & [\alpha^*(\mathcal{Y} \times \mathbb{A}^1), \mathcal{X}]_{s,et} \\
& & \wr\wr \\
& \xrightarrow[\substack{\text{since} \\ \mathcal{X} \text{ is } \mathbb{A}^1\text{-local} \\ \text{in et top.}}]{\simeq} & [(\alpha^* \mathcal{Y}) \times \mathbb{A}^1, \mathcal{X}]_{s,et}
\end{array}$$

Thus $B_{et} G := \alpha_*(BG^f)$ is $\mathbb{A}^1$-local in Spc_k.

Conversely, Let $B_{et} G$ be $\mathbb{A}^1$-local.

$\mathcal{X} \in Spc_k$

$$\begin{array}{ccc} [(\mathcal{X}, x), (B_{et} G, *)]_s & \xrightarrow{\simeq ?} & [(\mathcal{X}, x), (B_{et} G, *)]_{\mathbb{A}^1} \\ \wr\| \; (\alpha^*, \alpha_*) & & \\ [\alpha^*(\mathcal{X}, x), (BG^f, *)]_{s, et} & & \end{array}$$

$\therefore$ For $(\mathcal{X}, x) = \Sigma^i_s \wedge U_+$ we get

$$\begin{array}{ccc} [\Sigma^i_s \wedge U_+, (BG, *)]_{H^{et}_{\mathbb{A}^1, \cdot}(k)} & \longrightarrow & [\Sigma^i_s \wedge U_+, (B_{et} G, *)]_{\mathbb{A}^1} \\ \wr\| \; \text{Prop. 1.16, Page 72, MV} & & \\ H^{1-i}_{et}(U, G) & & \end{array}$$

Thm (Morel)**:** Let $(\mathcal{X}, x)$ be a pointed space. Then $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is strongly $\mathbb{A}^1$-invariant for all $i > 0$.

Corollary: Let G be strongly $\mathbb{A}^1$-invariant in etale topology. Then G is strongly $\mathbb{A}^1$-invariant in Nisnevich topology.

Proof: We have $B_{et} G$ is $\mathbb{A}^1$-local.

So $\pi_1^s(B_{et} G, *) \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(B_{et} G, *)$ is an iso of sheaves.

For each $U \in Sm/k$,

$$[\Sigma^1_s \wedge U_+, (B_{et} G, *)]_{\mathbb{A}^1} \simeq H^0_{et}(U, G) = G(U)$$

Thus taking the Nisnevich sheafification of both the presheaves

strongly $\mathbb{A}^1$-inv. sheaf by Morel $\rightarrow$ $\pi_1^{\mathbb{A}^1}(B_{et}G, *) \simeq a_{Nis}(G) = G$

$\therefore$ G is strongly $\mathbb{A}^1$-inv in Nisnevich topology.

Category of strongly $\mathbb{A}^1$-invariant sheaves

$$\begin{tikzcd} & \mathrm{Spc.} \arrow[d, shift right] & S \arrow[d, mapsto] \\ \text{Nis sheaf of gps} = Gr_k \arrow[r, shift left] & Gr_k^{\mathbb{A}^1} \arrow[u, shift right] \arrow[l, hook, shift left] & \pi_1^{\mathbb{A}^1}(\Sigma^1_s S, *) = F_{\mathbb{A}^1}(S). \end{tikzcd}$$

$$G \longmapsto \pi_1^{\mathbb{A}^1}(BG, *)$$

Pf: We have adjoint pair

$$G \longmapsto \pi_1^{\mathbb{A}^1}(BG, *)$$

$$Gr_k \rightleftarrows Gr_k^{\mathbb{A}^1}$$

$$\begin{tikzcd} {[(BH,*),(BG,*)]_s} \arrow[r, "G\text{-strongly } \mathbb{A}^1\text{-inv}", "\simeq"'] \arrow[d, "\pi_1^s(-)"'] & {[(BH,*),(BG,*)]_{\mathbb{A}^1}} \arrow[d, "\pi_1^{\mathbb{A}^1}(-)"] \\ \mathrm{Hom}_{Gr_k}(H, G) & \mathrm{Hom}_{Gr_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(BH,*), G) \end{tikzcd}$$

Right vertical map is surjective: Given $\pi_1^{\mathbb{A}^1}(BH,*) \xrightarrow{\phi} G$

$$\begin{tikzcd} & K(\pi_1^{\mathbb{A}^1}(BH),1) \arrow[dr, "\simeq"] & \\ BH \arrow[ur, dashed, "\exists"] \arrow[rr] & & P^{(1)}(BH) \arrow[d] \\ & & P^{(0)}(BH) \simeq * \end{tikzcd}$$

$$\begin{tikzcd} K(\pi_1^{\mathbb{A}^1}(BH,*),1) \arrow[r, "\exists"] & K(G,1) \arrow[d, equal] \\ BH \arrow[u, "\exists"] \arrow[r, dashed] & BG \end{tikzcd}$$

$\Rightarrow \exists f$ s.t. $\pi_1^{\mathbb{A}^1}(f) = \phi$.

Right vertical map is injective: Let $(BH,*) \overset{f}{\underset{g}{\rightrightarrows}} (BG,*)$ s.t.
$f_* = g_* : \pi_1^{\mathbb{A}^1}(BH,*) \to G$

$\Rightarrow f = g : BH \longrightarrow K(\pi_1^{\mathbb{A}^1}(BH),1) \overset{f_*}{\underset{g_*}{\rightrightarrows}} K(BG,1) \xrightarrow{\sim} BG$

Left vertical map is surj: Given $H \xrightarrow{\phi} G$, let $BH \xrightarrow{B(\phi)} BG$

then on $\pi_1^s(-)$:

$$\begin{array}{ccc} \pi_1^s(BH) & \longrightarrow & \pi_1^s(BG) \\ \wr\!\wr & & \wr\!\wr \\ H & \xrightarrow{\phi} & G \end{array}$$

Inj:

$$BH \xrightarrow{\sim} K(\underbrace{\pi_1^s(BH)}_{H},1) \underset{\rightarrow}{\overset{\rightarrow}{=}} K(\underbrace{\pi_1^s(BG)}_{G},1) \cong BG$$

<u>Lemma:</u> The category $Gr_k^{\mathbb{A}^1}$ admits small colimits.

We have adjoint pair $Gr_k \rightleftarrows Gr_k^{\mathbb{A}^1}$

Given a colimit diagram $\{H_i\}$ in $Gr_k^{\mathbb{A}^1}$,

the $\operatorname{colim}_i H_i$ exists in Gr_k.

Since $\pi_1^{\mathbb{A}^1}(B(-),*) : Gr_k \longrightarrow Gr_k^{\mathbb{A}^1}$ being left adjoint preserves colimits

Hence $\pi_1^{\mathbb{A}^1}(B(\operatorname{colim}_i H_i)) \simeq \operatorname{colim}_i \pi_1^{\mathbb{A}^1}(BH_i)$.

<u>Def:</u> Given $G_1 \leftarrow H \rightarrow G_2$

diagram of strongly $\mathbb{A}^1$-inv sheaves of groups:

$$G_1 *_H^{\mathbb{A}^1} G_2 := \pi_1^{\mathbb{A}^1}(B(G_1 *_H G_2))$$

"amalgamated sum of G_1 and G_2 over H."

Used later in "Van-Kampen thm" in $\mathbb{A}^1$-homotopy theory.

$G_1 *_H G_2$ = colimit in Gr_k.

Def: The free strongly $\mathbb{A}^1$-inv sheaf of groups on a pointed sheaf of sets (S, s) is

$$F_{\mathbb{A}^1}(S) = \pi_1^{\mathbb{A}^1}(\Sigma^1_s S).$$

For any strongly $\mathbb{A}^1$-invariant sheaf of groups G

$$\mathrm{Hom}_{Gr_k^{\mathbb{A}^1}}(F_{\mathbb{A}^1}(S), G) \xrightarrow{\sim} \mathrm{Hom}_{Spc_{\bullet,k}}((S,s), G).$$

pf: BG is $\mathbb{A}^1$-local $\Rightarrow$

$$\begin{tikzcd}
{[\Sigma^1_s S, BG]_s} \arrow[r, "\simeq"] \arrow[d, "\pi_1^{\mathbb{A}^1}(-)"', "\wr"] \arrow[dr, "\sim"] & {[\Sigma^1_s S, BG]_{\mathbb{A}^1}} \\
\mathrm{Hom}_{Gr_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(\Sigma^1_s S), G) & {[(S,s), \Omega^1_s BG]_s} \arrow[d, "\wr"] \\
 & \mathrm{Hom}_{Spc_{\bullet,k}}((S,s), G)
\end{tikzcd}$$

$G \underset{H_{s,\bullet}(k)}{\simeq} \Omega^1_s BG$

$$\begin{tikzcd}
 & K(\pi_1^{\mathbb{A}^1}(\Sigma^1 S), 1) \arrow[r] \arrow[d] & K(G,1) = BG \arrow[d] \\
\Sigma^1 S \arrow[r] \arrow[ur, dashed] & P^{(1)}(\Sigma^1 S) \arrow[r] \arrow[d] & P^1(BG) \\
 & * &
\end{tikzcd}$$

Proposition: Let $X \in Sm_k$ be $\mathbb{A}^1$-connected and G be an etale sheaf of groups strongly $\mathbb{A}^1$-invariant in etale topology.
For two points $x_1, x_2 \in X(k)$, the maps coincide i.e.

$$x_1^* = x_2^* : H^1_{et}(X, G) \longrightarrow H^1_{et}(\mathrm{Spec}\,k, G).$$

Let us denote this map by ρ for any rational point.
The natural map

$$\begin{tikzcd} & H^1_{et}(X, G) \arrow[r, "\rho"] & H^1_{et}(\mathrm{Spec}\,k, G) \\ H^1_{Nis}(X, G) \arrow[ur] \arrow[r, hook] & \rho^{-1}(*) \arrow[u, phantom, "\cup" description] & \end{tikzcd}$$

Proof: 1st part:

$$\begin{tikzcd} H^1_{et}(X, G) \arrow[r] & H^1_{et}(\mathrm{Spec}\,k, G) \\ {[X, B_{et}G]_{\mathbb{A}^1}} \arrow[u, "\wr"'] & {[\mathrm{Spec}\,k, B_{et}G]_{\mathbb{A}^1}} \arrow[u, "\wr"'] \\ {[X, B_{et}G]_s} \arrow[u, "\wr\ B_{et}G \text{ is } \mathbb{A}^1\text{-fibrant}"'] \arrow[ur, dashed] & \end{tikzcd}$$

$\tau \in [X, B_{et}G]_s$

Choose a representative $\tau : X \to B_{et}G$. The composite map

$$\begin{tikzcd} \mathrm{Spec}\,k \arrow[r, "x_i"] \arrow[d, equal] & X \arrow[r, "\tau"] \arrow[d] \arrow[dr] & B_{et}G \arrow[d] \\ * = \pi_0^{\mathbb{A}^1}(X) \arrow[r] & \pi_0^{\mathbb{A}^1}(X) \arrow[r, "\pi_0^{\mathbb{A}^1}(\tau)"'] & \pi_0^{\mathbb{A}^1}(B_{et}G) \end{tikzcd}$$

factors through $X \to \pi_0^{\mathbb{A}^1}(X)$.

Composing τ with $\mathrm{Spec}\,k \xrightarrow{x_i} X$ induces the same section $* \to \pi_0^{\mathbb{A}^1}(B_{et}G)$

i.e. an elt in $\pi_0^{\mathbb{A}^1}(B_{et}G)(k) = H^1_{et}(k, G)$.

$$\tau \longmapsto \pi_0^{\mathbb{A}^1}(\tau)$$

Hence independent of $x_i \in X(k)$.

2nd part: Thinking of G as Nisnevich sheaf, BG is $\mathbb{A}^1$-0-connected.

The map $(BG, *) \longrightarrow (B_{et}G, *)$

induces inclusion $* = \pi_0^{\mathbb{A}^1}(BG) \longrightarrow \pi_0^{\mathbb{A}^1}(B_{et}G)$

& iso $\pi_i^{\mathbb{A}^1}(BG) \longrightarrow \pi_i^{\mathbb{A}^1}(B_{et}G)$.

Thus $(BG, *) \to (B_{et}G, *)$ is the inclusion of the $\mathbb{A}^1$-connected component of the base point.

We have:

$$p^{-1}(*) = \left\{ X \to B_{et}G \;\middle|\; \begin{array}{ccc} X & \to & B_{et}G \\ \downarrow & \exists & \downarrow \\ \pi_0^{\mathbb{A}^1}(X) & \xrightarrow{*} & \pi_0^{\mathbb{A}^1}(B_{et}G) \end{array} \right\}$$

$* =$ Conn. component of $*$.

$$\begin{tikzcd}
 & p^{-1}(*) \arrow[d] & \\
{[X, BG]_{\mathbb{A}^1}} \arrow[r] \arrow[ur, dashed, "\exists \text{ as } \pi_0^{\mathbb{A}^1}(BG) = *"] \arrow[d, "\wr\ BG \text{ is } \mathbb{A}^1\text{-local}"] & {[X, B_{et}G]_{\mathbb{A}^1}} \arrow[r, "p"] \arrow[d, "\wr"] & {[\operatorname{Spec} k, B_{et}G]_{\mathbb{A}^1}} \\
H^1_{Nis}(X, G) \arrow[r] & H^1_{et}(X, G) &
\end{tikzcd}$$

$[\operatorname{Spec} k, B_{et}G]_{\mathbb{A}^1} = \pi_0^{\mathbb{A}^1}(B_{et}G)(\operatorname{Spec} k)$

Claim: $[X, BG]_{\mathbb{A}^1} \longrightarrow p^{-1}(*)$ is bijective.

Clearly 1-1.

Onto: Choose a representative $\tau : X \to B_{et}G$ s.t.

$$\begin{array}{ccc} X & \xrightarrow{\tau} & B_{et}G \\ \downarrow & & \downarrow \\ \pi_0^{\mathbb{A}^1}(X) & \xrightarrow{*} & \pi_0^{\mathbb{A}^1}(B_{et}G) \end{array}$$

inclusion of connected component of $*$.

X is $\mathbb{A}^1$-connected, so has a Postnikov tower

$$\begin{tikzcd}
 & K(\pi_1^{\mathbb{A}^1}(X,*),1) \arrow[r, "\exists"] \arrow[d] & K(\pi_1^{\mathbb{A}^1}(B_{et}G,*)) \simeq K(G,1) = BG \arrow[d] \\
X \arrow[ur, "\exists"] & P^{(1)}X \arrow[r] \arrow[d] & P^{(1)}\left(B_{et}G^{(0)}\right) \\
 & X &
\end{tikzcd}$$

$\Big\}$ functoriality of Postnikov tower

<u>Proposition:</u> Let k be a separably closed field of exp. char p. If X is $\mathbb{A}^1$-connected, then X admits no non-trivial finite etale Galois covers of degree coprime to p.

Hence $\pi_1^{et}(X,x)$ is pro-p-group.

<u>Pf:</u> Let $Y \to X$ be a finite etale Galois cover associated to a group G of order coprime to p.

Since G is an etale group scheme of order coprime to p.

by [MV, §4, Prop 3.1, page 137], $B_{et}G$ is $\mathbb{A}^1$-local.

Thus G is strongly $\mathbb{A}^1$-invariant in etale topology.

Hence by above proposition

$$H^1_{Nis}(X,G) \xrightarrow[\simeq]{} H^1_{et}(X,G) \to H^1_{et}(\operatorname{Spec} k, G) \cong *$$

k sep. closed

$\therefore$ $Y \xrightarrow{f} X$ is Nisnevich locally trivial G-torsor.

Claim: $Y \xrightarrow{f} X$ admits a section i.e. $\exists\, s : X \to Y$ s.t. $f \circ s = \mathrm{id}$.

WLOG we can assume that X is irreducible smooth variety $/k$.

By assumption that $Y \to X$ is Nis. locally trivial

i.e.
$$\begin{tikzcd}
G \times W \arrow[r, "\sim"] \arrow[dr, "pr_2"'] & W \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "et"] \\
& W \arrow[r, "p", "Nis"'] & X \\
& p^{-1}(U) \arrow[r] \arrow[u] \arrow[uul, bend left, "e \times id"] & U \arrow[u, hook, "\text{open dense}"'] \arrow[l, bend left]
\end{tikzcd}$$

Thus
$$\begin{tikzcd}
U \times G \arrow[r] \arrow[d] & Y \arrow[d, "f\ et"] \\
U \arrow[r, hook, "\text{open dense}"'] \arrow[u, bend left, "\exists s"] & X
\end{tikzcd}$$

Consider

$$\Gamma_s \hookrightarrow U \times_X Y \quad \text{the graph of } s$$

& $\overline{\Gamma_s}$ the closure of Γ_s in Y.

$$\begin{tikzcd}
\Gamma_s \arrow[r, hook] \arrow[d, hook] & \overline{\Gamma_s} \arrow[d, "\text{closed}"] \arrow[dd, bend left, "p"] \\
U \times_X Y \arrow[r, hook, "o"] \arrow[d] & Y \arrow[d, "fin"] \\
U \arrow[r, hook, "o"] & X
\end{tikzcd}$$

$\overline{\Gamma_s}$ = reduced closed subscheme

Note
$$\begin{tikzcd}
\Gamma_s \arrow[r, hook, "o"] \arrow[d, "iso"] & \overline{\Gamma_f} \arrow[d, "p"] \\
U \arrow[r, hook, "o"] \arrow[u, bend left, "s"] & X
\end{tikzcd}$$

Then $p : \overline{\Gamma_f} \to X$ is such that $p|_{\Gamma_s} : \Gamma_s \to U$ is iso.

$\Rightarrow$ p is birational.

Further p is finite, hence proper.

Thus $p : \overline{\Gamma_s} \longrightarrow X$ is a birational, proper map of integral schemes

$\Rightarrow$ each fiber of p is connected.

but p is finite, hence quasi-finite.

$\Rightarrow$ each fiber is singleton.

Also p is proper, hence closed, & $p(\overline{\Gamma_s}) \supset U$ dense in X.

$\therefore \quad p(\overline{\Gamma_s}) = X$

i.e. p is surjective.

Thus $p : \overline{\Gamma_s} \longrightarrow X$ is proper birational bijective map $\Rightarrow$ p is iso.

A. Mathew's Zariski's main thm application Cor. 3.9

Consider
$$\begin{tikzcd} \overline{\Gamma_s} \arrow[d, hook, "i"] \arrow[dd, bend left=60, "p"] \\ Y \arrow[d, "f"] \\ X \arrow[uu, bend left=60, "p^{-1}"] \end{tikzcd}$$

Thus $\underset{\overset{\shortparallel}{g}}{i \circ p^{-1}} : X \to \overline{\Gamma_s} \to Y$ s.t. $f \circ g = id_X$

Hence $\pi_1^{et}(X, x)$ is pro-p-group.

Strict $\mathbb{A}^1$-invariance and $\mathbb{A}^1$-local Eilenberg-MacLane spaces

Def: Let A be a Nisnevich sheaf of abelian groups.
A is strictly $\mathbb{A}^1$-invariant if $\forall\ U \in Sm_k$

$$H^i_{Nis}(U, A) \xrightarrow{\simeq} H^i_{Nis}(U \times \mathbb{A}^1, A) \quad \forall\ i \geq 0.$$

Given etale sheaf A, A is strictly $\mathbb{A}^1$-inv in etale topology if

$$H^i_{et}(U, A) \xrightarrow{\sim} H^i_{et}(U \times \mathbb{A}^1, A) \quad \forall\ i \geq 0.$$

A is strictly $\mathbb{A}^1$-inv $\Leftrightarrow$ $K(A,i)$ is $\mathbb{A}^1$-local $\forall i \geq 0$

A is strictly $\mathbb{A}^1$-inv in et top $\Leftrightarrow$ $K(A,i)$ is $\mathbb{A}^1$-local $\forall i \geq 0$ in Spc_k^{et}.

Def: Let A be etale sheaf $\quad K(A,i) \in Spc_k^{et}$.

Let $K(A,i)^f$ = fibrant replacement of $K(A,i)$ in Spc_k^{et}.

Define $\quad K_{et}(A,i) = \alpha_* K(A,i)^f \in Spc_k$.

Lemma: Let A be etale sheaf of abelian groups. Then
A is strictly $\mathbb{A}^1$-inv in etale topology iff $K_{et}(A,i)$ is $\mathbb{A}^1$-local $\forall i \geq 0$.
Moreover if A is strictly $\mathbb{A}^1$-inv etale top, then $\forall\ U \in Sm_k$

$$[\Sigma^j_s \wedge U_+, K_{et}(A,i)]_{\mathbb{A}^1} \xrightarrow{\sim} H^{i-j}_{et}(U, A)$$

$\forall\ 0 \leq j \leq i$.

Pf: Similar as before:

$$\begin{tikzcd}
{[\Sigma^j_s \wedge U_+, K_{et}(A,i)]_{\mathbb{A}^1}} \arrow[r, "\simeq", "\text{adjunction}"'] \arrow[dddr, "\simeq"'] & {[\alpha^*(\Sigma^j_s \wedge U_+), K(A,i)^f]_{H^{et}_{\mathbb{A}^1}(k)}} \arrow[d, "\wr"] \\
& {[\Sigma^j_s \wedge U_+, K(A,i)^f]_{H^{et}_{\mathbb{A}^1}(k)}} \arrow[d, "\wr\ K(A,i) \text{ is } \mathbb{A}^1\text{-local}"] \\
& {[\Sigma^j_s \wedge U_+, K(A,i)]_{H^{et}_s(k)}} \arrow[d, "\wr"] \\
& H^{i-j}_{et}(U, A)
\end{tikzcd}$$

book: Thm 5.46

Thm (Morel): Let A be strongly $\mathbb{A}^1$-invariant sheaf of abelian groups. Then A is strictly $\mathbb{A}^1$-invariant.

Pf: (Sketch)

$$\left.\begin{array}{l} H^*(U, A) \simeq H^*(C^\cdot_{RS}(U; A)) \\ C^\cdot_{RS}(U; A) \xrightarrow[\text{quasi-isomorphism}]{} C^\cdot_{RS}(U \times \mathbb{A}^1; A) \end{array}\right\} \forall U \in Sm_k$$

Cor: Let A be an etale sheaf of abelian groups strongly $\mathbb{A}^1$-invariant in et. top. Then the underlying Nisnevich sheaf A is strictly $\mathbb{A}^1$-invariant.

Pf:

$$\begin{array}{ccc} \text{strong } \mathbb{A}^1\text{-inv in et top} & \Rightarrow & \text{strong } \mathbb{A}^1\text{-inv. in Nis-top} \\ & & \Downarrow \text{Thm} \\ & & \text{strict } \mathbb{A}^1\text{-inv.} \end{array}$$

Summarizing :

gives examples of

$$\begin{tikzcd} & Ab^{et}_{strict,\mathbb{A}^1} \arrow[r, phantom, "\subseteq"', "\text{full subcat}"] \arrow[d, hook] & Ab^{et}_{strong,\mathbb{A}^1} \arrow[d, hook] \\ \mathbb{Z}[1/p]\text{-mod}^{et,tr}_{\mathbb{A}^1\text{-inv}} \arrow[ur] & Ab^{Nis}_{strict,\mathbb{A}^1} \arrow[r, phantom, "\simeq"] & Ab^{Nis}_{strong,\mathbb{A}^1} \end{tikzcd}$$

Lemma: Let k be a field so that ~~ex~~ char $k = p$. Let A be etale sheaf $\mathbb{Z}[1/p]$-modules that is $\mathbb{A}^1$-invariant. Then A is strictly $\mathbb{A}^1$-invariant for etale topology.

Pf: Given a s.e.s. of etale sheaves of $\mathbb{Z}[1/p]$-modules

$$0 \to A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

if 2 of them are strictly $\mathbb{A}^1$-inv in et top then so is 3rd.

Consider $0 \to A_{tors} \to A \to A \otimes \mathbb{Q} \to A \otimes \mathbb{Q}/\mathbb{Z} \to 0$

Note A_{tors} an torsion coprime to p.

Reduce to show lemma for A either

1) A is a sheaf of $\mathbb{Q}$-vector spaces or
2) A is torsion coprime to p.

} etale sheaves with transfers.

Ref: Prop 14.23 lectures on motivic coh.

In case 1) $\forall i: H^i_{et}(U, A) \simeq H^i_{Nis}(U, A)$. & this in $\mathbb{A}^1$-inv.

Thm. 13.8

2) [MVW, Thm 7.20]: Suslin's rigidity thm :

A is locally constant : $\pi^* \pi_* A \xrightarrow{\sim} A$ is iso where

$$\pi^* : Shv_{et}(Et/_k) \rightleftarrows Sh_{et}(Sm_k) : \pi_*$$

Then by [SGA 4, Tome 3, XV, cor 2.2]

$$H^i_{et}(-, A) \text{ is } \mathbb{A}^1\text{-invariant}.$$

<u>Prop:</u> Let k be a field of exp char p. Let $X \in Sm/_k$.
Let $\ell \neq p$, $\mathbb{G}_m' = \mathbb{G}_m \overset{\cdot}{\otimes}_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$. Then $Br(X)[\ell] \hookrightarrow H^2_{et}(X, \mathbb{G}_m')$.

<u>Proof:</u> Let $\mathbb{G}_m'$ = etale sheaf <u>associated to</u> the presheaf
actually a sheaf

$$U \longmapsto \mathcal{O}(U)^\times \otimes \mathbb{Z}[1/p]$$

char $k=0$: $\mathbb{G}_m' = \mathbb{G}_m$

We know $Br(X) \hookrightarrow H^2_{et}(X, \mathbb{G}_m)$

We have diagram

$$\begin{tikzcd}
\overset{\alpha}{\underset{\cap}{}} Br(X) \arrow[r, hook, "\Psi"] \arrow[dr, dashed, "\Phi"] & H^2_{et}(X, \mathbb{G}_m) \arrow[d] \\
Br(X)[\ell] \arrow[u, hook] \arrow[r] & H^2_{et}(X, \mathbb{G}_m')
\end{tikzcd}$$

ℓ-torsion subgp

<u>Claim:</u> for ℓ coprime to p : ϕ is injective on $Br(X)[\ell]$.

For if $\alpha \in Br(X)[\ell]$ s.t. $0 = \phi(\alpha)$.

We have $\ell^n \cdot \alpha = 0$ in $Br(X)$ for some $n \geq 1$.

$\mathbb{G}_m'$ is $\mathbb{Z}[1/p]$-modules $\Rightarrow$ $H^2_{et}(X, \mathbb{G}_m')$ is also $\mathbb{Z}[1/p)$-modules.

So $\phi(\alpha) = 0 \Rightarrow \exists\, m$ s.t. $p^m \Psi(\alpha) = 0$ in $H^2_{et}(X, \mathbb{G}_m)$.

$\Psi(p^m \cdot \alpha) = 0 \implies p^m \alpha = 0$ in $Br(X)$.

But $\ell^n \cdot \alpha = 0$, since α in $Br(X)[\ell]$.

$\therefore$ $\alpha = 0$ in $Br(X)$

i.e. $\phi : Br(X)[\ell] \longrightarrow H^2_{et}(X, \mathbb{G}_m')$ is injective for $\ell \neq p$.

Prop. 4.3.8: k field exp. char $k = p$. $x \in X(k)$. X $\mathbb{A}^1$-connected.

Then $X \underset{x}{\overset{\pi}{\rightleftarrows}} \operatorname{Spec} k$ induces isomorphism

$$H^2_{et}(X, \mathbb{G}_m') \xrightarrow{\cong} H^2_{et}(k, \mathbb{G}_m').$$

In particular if k is separably closed, $Br(X)$ is p-group.

Pf:

$$\begin{tikzcd}
H^2_{et}(k, \mathbb{G}_m') \arrow[r, "\Rightarrow 1-1"] \arrow[d, phantom, "\wr"] \arrow[rr, bend left, "id"] & H^2(X, \mathbb{G}_m') \arrow[r] \arrow[d, phantom, "\wr"] & H^2_{et}(k, \mathbb{G}_m') \arrow[d, phantom, "\wr"] \\
{[S^0 \wedge \operatorname{Spec} k_+, K_{et}(\mathbb{G}_m', 2)]} \arrow[r] & {[X_+, K_{et}(\mathbb{G}_m', 2)]} \arrow[r] & {[\operatorname{Spec} k_+, K_{et}(\mathbb{G}_m', 2)]}
\end{tikzcd}$$

To show 1st map is onto:

$$\begin{tikzcd}
\text{Let } X \arrow[r, "\tau"] \arrow[d] & K_{et}(\mathbb{G}_m', 2) \arrow[d] \\
* = \pi_0^{\mathbb{A}^1}(X) \arrow[r, "\pi_0^{\mathbb{A}^1}(\tau)"] & \pi_0^{\mathbb{A}^1}(K_{et}(\mathbb{G}_m', 2))
\end{tikzcd}$$

$\therefore$ $\pi_0^{\mathbb{A}^1}(\tau) \in [\operatorname{Spec} k, K_{et}(\mathbb{G}_m', 2)]$

s.t. $\pi_0^{\mathbb{A}^1}(\tau) \longmapsto \tau \longmapsto \pi_0^{\mathbb{A}^1}(\tau)$

Thus $H^2_{et}(X, \mathbb{G}_m') \longrightarrow H^2_{et}(k, \mathbb{G}_m')$ is bijective.

$$\begin{tikzcd} Br(X) \arrow[r, hook] & H^2_{et}(X, \mathbb{G}_m) \arrow[d] \\ \text{for } \ell \neq p: Br(X)[\ell] \arrow[u, hook] \arrow[r, hook] & H^2_{et}(X, \mathbb{G}_m') \simeq H^2_{et}(k, \mathbb{G}_m') \end{tikzcd}$$

$H^2_{et}(k, \mathbb{G}_m') = 0$, k sep. closed.

$\therefore \; Br(X)[\ell] = 0$ for $\ell \neq p$.

$\therefore \; Br(X)$ is p-torsion.

<u>Example:</u> Let $X =$ K3 surface $/k$. $Br(X) \neq 0$. $\Rightarrow X$ is not $\mathbb{A}^1$-conn.

Let $k =$ alg closed field of char exponential p

Let X smooth proper$/k$.

Let $k = \mathbb{C}$:

$$0 \to \left(\frac{H^2(X, \mathbb{Z})}{Pic(X)}\right) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(X, \mathbb{G}_m) \to H^3(X, \mathbb{Z})_{tors} \to 0$$

is exact.

(this follows from $0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$) & then $Tor_1(-, \mathbb{Q}/\mathbb{Z})$ & note $Tor_1(M, \mathbb{Q}/\mathbb{Z}) = M_{tors}$

Thus $H^2(X, \mathbb{G}_m) \simeq (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho} \oplus H^3(X, \mathbb{Z})_{tors}$.
non-canonically

So if $b_2 > \rho$, (e.g. $X =$ K3 surface $b_2 = 22$ & $\rho = 20$)
X is not $\mathbb{A}^1$-connected.

for char $k > 0$, $\ell \neq p$

$$H^2_{et}(X, \mathbb{G}_m)[\ell] \simeq \left(\mathbb{Q}_\ell/\mathbb{Z}_\ell\right)^{b_2 - \rho} \oplus M$$

where M -- finite ℓ-group

Example: Let G = simply connected, semi-simple alg. group over a field k, exp char $k = p$.

$$G^+(k) = \underset{\text{gen.}}{\langle} f(\mathbb{G}_a(k)) \mid f : \mathbb{G}_a \longrightarrow G \text{ homomorphism} \rangle$$

$$G(k) \longrightarrow \frac{G(k)}{G^+(k)} = W(k, G) = \text{"Whitehead group"}$$

Kneser-Tits problem: For which groups G, $W(k, G) = 1$.

We say a group G is W-trivial if $W(L, G) = 1 \quad \forall L/k$.

Question (Gille) Can we characterize G s.t. G is W-trivial?

Note: G is W-trivial $\Rightarrow$ $\mathbb{A}^1$-chain connected $\Rightarrow$ $\mathbb{A}^1$-connected.

(Gille: in addition to above assumptions if G is split, then
G is W-trivial $\Leftrightarrow$ ~~$\Rightarrow$~~ hence $\mathbb{A}^1$-conn.)

By what we saw today, G-W-trivial $\Rightarrow$ $Br(G)$ is p-torsion.

END!