

Recall:

Def. The category \mathcal{S}_k -Set has the same objects
 (from \mathcal{S}_k -Set := $(\mathcal{S}_k, \text{Set})$ with the following additional data:
 $\forall (L, \nu)$ with $L \in \mathcal{S}_k$ and ν a double valuation on L
 such that K_ν is separable over k there is a unique map
 $(\mathcal{S}_k\text{-Set}) \ni S: S(L) \rightarrow S(K_\nu)$
 The morphisms in \mathcal{S}_k -Set are the nat. transf. preserving
 this add. data.

Now, the important thing to note is if (L, ν)
 is such a complete pair, denoting \mathcal{O}_L the d.v.a.
 the morphism $\mathcal{O}_L \rightarrow L$ induces
 $\text{Spec } \mathcal{O}_L \rightarrow \text{Spec } (L)$

which identifies $\text{Spec } (L)$ with an open and dense
 subscheme of $\text{Spec } (\mathcal{O}_L)$ from $\mathcal{O}_L \ni \nu$ v.v.a.
 Here, if $S \in \text{Sh}_k^{\text{an}}$, by bi-universality, the map
 $S(\text{Spec } \mathcal{O}_L) \rightarrow S(\text{Spec } L)$
 is a bijection. Furthermore, $\mathcal{O}_L \rightarrow k_\nu$ induces
 $S(\text{Spec } \mathcal{O}_L) \xrightarrow{\cong} S(\text{Spec } k_\nu)$

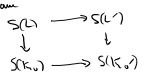
so we get a unique morphism
 $S(L) \xrightarrow{\cong \circ \tau_\nu^{-1}} S(K_\nu)$.

In other words, we obtain a function
 $\text{Sh}_k^{\text{an}} \rightarrow \mathcal{S}_k^{\text{an}}\text{-Set}$
 $S \mapsto \begin{matrix} \mathcal{S}_k \\ L \end{matrix} \mapsto S(\text{Spec } L)$

Returning to the indelible case and using
 the transit. property we get the full-faithfulness
 of this function.

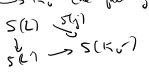
Def. An object $S \in \mathcal{S}_k^{\text{an}}\text{-Set}$ is called sheaflet if the
 following properties hold:

(A1). Given pairs (L, ν) and (L', ν') s.t.
 $L'/L \in \mathcal{S}_k$, ν' restricts to a d.v.a. ν on L
 w/ non-triv. inter L and $K_\nu, K_{\nu'}/k$ separable,
 the following diagram

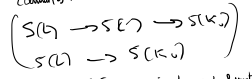


commutes.

(A2). Given pairs (L, ν) and L s.t. $L'/L \in \mathcal{S}_k$
 ν' restricts to ν on L and K_ν/k separable,
 $j: L \hookrightarrow K_\nu$ the following diagram



commutes.



(A3). $\forall X \in \text{Sh}_k^{\text{an}}$ ind. w/ function field E
 $\forall e \in X^{(1)}$ s.t. $K(e)/k$ sep. and $\nu_e \in X^{(2)}$
 s.t. $\left. \begin{matrix} e \in X \\ j_e \in \text{Sh}_k^{\text{an}} \end{matrix} \right\} \text{the composition } S(L) \rightarrow S(K(j_e)) \rightarrow S(K(e))$
 is independent of j_e .

If the last axiom

(A4) $\forall L \in \mathcal{O}_A$, the map $S(L) \rightarrow S(L \otimes L)$

is a bijection
 volds wir say that S is shufflelike and A^c -invariant.

Theorem The restriction functor $S_{\text{Shuffle}} \rightarrow \mathcal{O}_A^{\text{Shuffle}}$ set.

\square fully faithful and has essential image the full subcat. of $\mathcal{O}_A^{\text{Shuffle}}$ spanned by shufflelike and A^c -invariant objects.

\square Full faithfulness: OK.
 The es. image \subseteq full subcat. spanned by shufflelike and A^c -inv. objects:

(A4): OK.
 (A1): We have to show that the following diagram commutes.

$$\begin{array}{ccc} S(\text{Spec}(L)) & \longrightarrow & S(\text{Spec}(L')) \\ \downarrow \simeq & & \downarrow \simeq \\ S(\text{Spec}(D_U)) & \longrightarrow & S(\text{Spec}(D_{U'})) \\ \downarrow & & \downarrow \\ S(\text{Spec}(k_U)) & \longrightarrow & S(\text{Spec}(k_{U'})) \end{array}$$

Commutator
 The bottom sq. commutes by functoriality.

The top sq.: $\text{Spec}(L') \rightarrow \text{Spec}(D_{U'})$
 \downarrow
 $\text{Spec}(L) \xrightarrow{\text{isom}} \text{Spec}(D_U)$
isom. seen (open disc)

is a distinguished sq., hence, thanks to the Milnorick charact. via distinguished sq., the following diagram commutes.

$$\begin{array}{ccc} S(D_{U'}) & \longrightarrow & S(D_{U'}) \\ \downarrow & & \downarrow \\ S(L) & \longrightarrow & S(L') \end{array}$$

(A2) and (A3): choose some explicit smooth models

for appropriate closed immersions.

Construction of the quasi-inverse of the restriction functor:

Let $S \in \mathcal{O}_A^{\text{Shuffle}}$ set be shufflelike and A^c -invariant, so set a presheaf $\tilde{S} \in \text{Psh}(\text{Sm}_A)$ assigning to be S_{Shuffle} in $\mathcal{O}_A^{\text{Shuffle}}$
 $\tilde{S}(k(U)) = S(L)$ and forcing the first property to be true, c.e. $\forall X \times \text{Sm}_A$,
 or mult.comp. X_L the map $\tilde{S}(X) \rightarrow \prod_{i \in X} \tilde{S}(X_i)$ is a bijection.

Let $Y \xrightarrow{f} X$ a morphism of smooth schemes,

- If f is dominant, we define

$$\tilde{S}(f): \tilde{S}(X) \rightarrow \tilde{S}(Y) \text{ as,}$$

for mult.comp., we use field extension $k(Y)/k(X)$

we set $\tilde{S}(Y_i) \rightarrow \tilde{S}(X_i)$ and $\tilde{S}(X) \rightarrow \prod_{i \in X} \tilde{S}(X_i) \rightarrow \prod_{i \in Y} \tilde{S}(Y_i) \rightarrow \tilde{S}(Y)$.

- If f is a direct immersion, let N_f denote the normal bundle of f . We can consider the diagram

$$\begin{array}{ccc} \mathbb{P}(N_f) & \xrightarrow{c} & \mathbb{P}E_c(X) \\ \pi \downarrow & & \downarrow \bar{\pi} \\ Y & \hookrightarrow & X \end{array}$$

Note that π is Zariski locally trivial map w/ projective space fibres. So, by construction,

\tilde{S} only depends on function fields and since S is M^+ -invariant we have $\tilde{S}(\mathbb{P}(N_f)) = \tilde{S}(Y)$.

Since $\bar{\pi}$ is biart and proper, again because \tilde{S} only depends on function fields, we have

$$\tilde{S}(\mathbb{P}E_c(X)) = \tilde{S}(X) \text{ The map } \tilde{S} \sim$$

c is a closed immersion of codim ≤ 1 and there is a discrete val. arising from the blow-up and we use the residue art. data to get a map $\tilde{S}(X) = \tilde{S}(\mathbb{P}E_c(X))$

$$\downarrow \\ \tilde{S}(Y) = \tilde{S}(\mathbb{P}(N_f)) \sim$$

- If f is a general immersion we can increase it as follows:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \pi \downarrow & \searrow \psi_{X,Y} & \uparrow \pi \end{array}$$

and since $\pi \circ f$ is a direct immersion and π is dominant we can define $\tilde{S}(f)$ as

$$\tilde{S}(f) = \tilde{S}(f) \circ \tilde{S}(Y)$$

By construction \tilde{S} is biart. and M^+ -invariant. Perhaps so, harder to see because from the previous table, \tilde{S} is a Nisnevich sheaf.

Let's try to check its compatibility and functoriality. This is probably the following lemma:

Lemma (1.2.8)
 Given a comm. $Y' \hookrightarrow Y \hookrightarrow X$
 $\downarrow \quad \downarrow$
 $X' \hookrightarrow X$

(the diagram $\tilde{S}(X') \rightarrow \tilde{S}(X) \rightarrow \tilde{S}(Y) \rightarrow \tilde{S}(Y')$ commutes.)

Given a square $Z \hookrightarrow Y \hookrightarrow X$ the triangle $\tilde{S}(X) \rightarrow \tilde{S}(Y) \rightarrow \tilde{S}(Z)$ commutes.

Let (L, N) s.t. $L \in \mathcal{O}_X$ and $N \in \mathcal{O}_X$ sep. if X is a proper of f.t. over k , we can consider the \mathcal{O}_X -bet defined by

$$L \mapsto X(L) / \cong \begin{matrix} x_1, x_2 \in X(L) \\ x_1 \sim x_2 \\ \cong \\ \Rightarrow M^+ \text{ bet } X \text{ s.t.} \\ f(\emptyset) = x_1, f(\emptyset) = x_2 \end{matrix}$$

and, thanks to the red. criterion of properness, the L -pts of X naturally extend to \mathbb{A}^1 -pts of X , i.e. we know that, the map $X(0_0) \rightarrow X(L)$ is a bijection.

$$\begin{pmatrix} \text{Spec}(A) & \xrightarrow{\sim} & X \\ \downarrow & \text{...} & \downarrow \\ \text{Spec}(0_0) & \xrightarrow{\sim} & \text{Spec}(k) \end{pmatrix}$$

Prop. 0
 If X is a proper scheme of f.t. over k , then,
 1. There is a factorization $X(L) = X(0_0) \rightarrow X(k_0) \rightarrow X(k_0)/\sim$
 where we get a $\tilde{\mathcal{A}}_k^e$ -set via the association $L \mapsto X(L)/\sim$

2. The object obtained by the previous pt, denoted $\tilde{w}_{\text{Nat}}(X)$, is sheaflike and \mathbb{A}^1 -invariant.

In order to prove the prop. 0 we use the following results:

Lemma 1. Let S be an inv. or smooth k -scheme of dim. 2 w/ function field F , then \exists finitely many closed pts $z_1, \dots, z_n \in S$.

Thanks to prop. 0 we can state the following result:
 Theorem. Suppose X is a proper scheme of f.t. over k .
 There exists a natural and \mathbb{A}^1 -invariant sheaf $\tilde{w}_{\text{Nat}}(X)$ (together w/ a map $X \rightarrow \tilde{w}_{\text{Nat}}(X)$ s.t. $\forall L \in \mathcal{A}_k$, there is a factorization $X(L) \rightarrow \tilde{w}_{\text{Nat}}(X)(L) \rightarrow X(0_0)/\sim$

pf. Using prop. 0, we define $\tilde{w}_{\text{Nat}}(X)$ as the sheaf of \tilde{w}_{Nat} associated to $\tilde{w}_{\text{Nat}}(X)$ via the equiv. of cat. given by the restriction functor. It remains to construct the map $X \rightarrow \tilde{w}_{\text{Nat}}(X)$.

We do it at the level of presheaves: if $U \in \mathcal{A}_k$ we can look at $X(U)/\sim$ the homotopy relation (i.e. $U \xrightarrow{\sim} X$ via homotopy

$$\begin{pmatrix} U & \xrightarrow{\sim} & X \\ \downarrow & \text{...} & \downarrow \\ \text{Spec}(k) & \xrightarrow{\sim} & \text{Spec}(k) \end{pmatrix}$$

and since this association is functorial there is a well-defined map $X(U) \rightarrow X(U)/\sim \rightarrow \tilde{w}_{\text{Nat}}(X)(U)$ induced by $\text{Spec}(k[U]) \rightarrow U \rightarrow \text{Spec}(k)$

In part., if $U = \text{Spec}(L)$ we get the bijection:

Finally, Same hypothesis on X , there is a canonical map Prop. $\tilde{w}_{\text{Nat}}(X) \rightarrow \tilde{w}_{\text{Nat}}(X) \simeq 1$.
 $\tilde{w}_{\text{Nat}}(X) \xrightarrow{\sim} \tilde{w}_{\text{Nat}}(X)$ induces a bijection $\tilde{w}_{\text{Nat}}(X)(L) \simeq \tilde{w}_{\text{Nat}}(X)(L)$ $\forall L \in \mathcal{A}_k$.

Pf. Since $\pi_0^{b\mathbb{A}^1}(X)$ is biinvt. and H^1 -inv. by construction) it implies that $\pi_0^{b\mathbb{A}^1}(X)$ is \mathbb{A}^1 -local (Moeil - Voevodsky char. of H^1 -loc. objects)

Hence the univ. prop. of the \mathbb{A}^1 -localization implies that the previous morphism

$$X \rightarrow \pi_0^{b\mathbb{A}^1}(X)$$

factors uniquely through the \mathbb{A}^1 -loc.:

$$\begin{array}{ccc} X & \rightarrow & \pi_0^{b\mathbb{A}^1}(X) \\ & \searrow & \uparrow \\ & & L_{\mathbb{A}^1}(X) \end{array}$$

Then, we have, $\forall u \in S_{\text{sm}}$, a morphism

$$[u, L_{\mathbb{A}^1}(X)]_S \rightarrow [u, \pi_0^{b\mathbb{A}^1}(X)]_S$$

which is functorial in u . But, since $\pi_0^{b\mathbb{A}^1}(X)$ is of simp. dim. 0 we have

$$[u, \pi_0^{b\mathbb{A}^1}(X)]_S = \pi_0^{b\mathbb{A}^1}(X)(u).$$

Hence, by def. of $\pi_0^{\mathbb{A}^1}(X)$, applying sheafification we obtain a morphism $\pi_0^{\mathbb{A}^1}(X) \rightarrow \pi_0^{b\mathbb{A}^1}(X)$

from the previous morphism

Finally, the morphism $X \rightarrow L_{\mathbb{A}^1}(X)$ induces

the morphism $X \rightarrow \pi_0^{\mathbb{A}^1}(X)$ which factors as

$$\begin{array}{ccc} X & \rightarrow & \pi_0^{\mathbb{A}^1}(X) \\ & \searrow & \uparrow \\ & & \pi_0^{\mathbb{A}^1}(X) \end{array}$$

In particular, we'll get a bijection between

the L -pts of $\pi_0^{\mathbb{A}^1}(X)$ and $\pi_0^{b\mathbb{A}^1}(X)$, i.e.

we proved the thm 2-4-3.