$\mathbb{A}^1$- chain connectedness and $\mathbb{A}^1$- connectedness

Recall:

<u>Def.</u> The category $\tilde{\mathcal{F}}_k^{\wedge}$ - Set has the same objects than $\mathcal{F}_k$ - Set $:= [\mathcal{F}_k, \mathrm{Set}]$ with the following <u>additional</u> data:
$\forall (L, v)$ with $L \in \tilde{\mathcal{F}}_k$ and v a discrete valuation on L such that K_v is separable over k there is a <u>residue morphism</u>

$(S \in \tilde{\mathcal{F}}_k\text{-Set})$ $\quad s_v :$ $S(L) \longrightarrow S(K_v)$

The morphisms in $\tilde{\mathcal{F}}_k^{\wedge}$ - Set are the nat. transf. preserving this add. data.

Now, the important thing to note is if (L, v) is such a couple, then, denoting $\mathcal{O}_v$ the d.v.r the morphism $\mathcal{O}_v \longrightarrow L$ induces

$$\mathrm{Spec}(L) \longrightarrow \mathrm{Spec}(\mathcal{O}_v)$$

which identifies $\mathrm{Spec}(L)$ with an open and dense subscheme of $\mathrm{Spec}(\mathcal{O}_v)$ since $\mathcal{O}_v$ is a d.v.r.
Hence, if $S \in Shv_k^{bA^1}$, by birationality, the map

$$S(\mathrm{Spec}\,\mathcal{O}_v) \xrightarrow{\varphi_v} S(\mathrm{Spec}\,L)$$

is a bijection. Furthermore, $\mathcal{O}_v \to K_v$ induces

$$S(\mathrm{Spec}\,\mathcal{O}_v) \xrightarrow{\pi_v} S(\mathrm{Spec}\,K_v)$$

so we get a residue morphism

$$S(L) \xrightarrow{\pi_v \circ \varphi_v^{-1}} S(K_v).$$

In other words, we obtain a functor

$$Shv_k^{bA^1} \longrightarrow \tilde{\mathcal{F}}_k^{\wedge}\text{-Set}$$

$$S \longmapsto \underset{L}{\overset{\mathcal{F}_k}{\cup}} \longmapsto S(\mathrm{Spec}\,L)$$

Reducing to the irreducible case and using the birat. property we get the full-faithfulness of this functor.

<u>Def.</u> An object $S \in \tilde{\mathcal{F}}_k^{\wedge}$-Set is called <u>sheaflike</u> if the following properties hold:

(<u>A1</u>). Given pairs (L, v) and (L', v') s.t. $L'/L \in \tilde{\mathcal{F}}_k$, v' restricts to a d.v.r v on L w/ ram. index 1 and $K_v, K_{v'}/k$ separable, the following diagram

$$\begin{array}{ccc} S(L) & \longrightarrow & S(L') \\ \downarrow & & \downarrow \\ S(K_v) & \longrightarrow & S(K_{v'}) \end{array}$$

commutes.

(<u>A2</u>). Given pairs (L', v') and L s.t. $L'/L \in \tilde{\mathcal{F}}_k$ v' restricts to 0 on L and $K_{v'}/k$ separable, $j : L \hookrightarrow K_{v'}$ the following diagram

$$\begin{array}{ccc} S(L) & \xrightarrow{S(j)} & \\ \downarrow & \searrow & \\ S(L') & \longrightarrow & S(K_{v'}) \end{array}$$

commutes.

$$\left(\begin{array}{l} S(L) \longrightarrow S(L') \longrightarrow S(K_{v'}) \\ S(L) \longrightarrow S(K_v) \end{array} \right)$$

(<u>A3</u>). $\forall X \in Sm_k$ <u>irred.</u> w/ function field F, $\forall z \in X^{(2)}$ s.t. $K(z)/k$ sep. and $\forall y_0 \in X^{(1)}$ s.t. $\left| \begin{array}{l} z \in \bar{y}_0 \\ \bar{y}_0 \in Sm_k \end{array} \right.$ the composition

$$S(F) \longrightarrow S(K(y_0)) \longrightarrow S(K(z))$$

is independent of y_0.

If the last axiom

(A4.) $\forall L \in \mathfrak{F}_A$, the map

$$S(L) \longrightarrow S(L(t))$$

is a bijection

holds we say that S is sheaflike and

$\mathbb{A}^1$-invariant.

Theorem The restriction functor

$$Shv_A^{\mathbb{A}^1} \longrightarrow \tilde{\mathfrak{F}}_A^{\wedge}\text{-Set}.$$

is fully faithful and has essential image the full subcat. of $\tilde{\mathfrak{F}}_A^{\wedge}$-Set spanned by sheaflike and $\mathbb{A}^1$-invariant objects.

Pf. • Full faithfulness: OK.

• The ess. image $\subseteq$ full subcat. spanned by sheaflike and $\mathbb{A}^1$-inv. objects:

- (A4.): OK.
- (A1.): We have to show that the following diagram

$$\begin{tikzcd}
S(\mathrm{Spec}(L)) \arrow[r] \arrow[d, "\sim"'] & S(\mathrm{Spec}\, L') \arrow[d, "\sim"] \\
S(\mathrm{Spec}\, \mathcal{O}_v) \arrow[r] \arrow[d] & S(\mathrm{Spec}\, \mathcal{O}_{v'}) \arrow[d] \\
S(\mathrm{Spec}\, \kappa_v) \arrow[r] & S(\mathrm{Spec}\, \kappa_{v'})
\end{tikzcd}$$

commutes.

The bottom sq. commutes by functoriality.

The top sq.:

$$\begin{tikzcd}
\mathrm{Spec}(L') \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{v'}) \arrow[d, "\text{étale}"] \\
\mathrm{Spec}(L) \arrow[r, hook, "\substack{\text{already} \\ \text{seen (open dense)}}"'] & \mathrm{Spec}(\mathcal{O}_v)
\end{tikzcd}$$

(essentially because the ram. index $= 1$)

is a distinguished sq., hence, thanks to the Nisnevich charact. via distinguished sq., the following diagram

$$\begin{tikzcd}
S(\mathcal{O}_v) \arrow[r] \arrow[d] & S(\mathcal{O}_{v'}) \arrow[d] \\
S(L) \arrow[r] & S(L')
\end{tikzcd}$$

commutes.

- (A2). and (A3): choose some explicit smooth models for appropriate closed immersions.

• Construction of the quasi-inverse of the restriction functor:

Let $S \in \tilde{\mathfrak{F}}_A^{\wedge}$-Set be sheaflike and $\mathbb{A}^1$-invariant, we get a presheaf $\tilde{S} \in \mathrm{PSh}(Sm_A)$ assigning to $U \in Sm_A$ irred. $\tilde{S}(k(U)) =: \tilde{S}(U)$ and forcing the first property to be true, i.e. $\forall X \in Sm_A$, w/ irred. comp. X_i the map

$$\tilde{S}(X) \longrightarrow \prod_{i \in X^{(0)}} \tilde{S}(X_i) \text{ is a bijection}$$

Let $Y \xrightarrow{f} X$ a morphism of smooth schemes,

- If f is dominant, we define

$$\tilde{S}(f): \tilde{S}(X) \longrightarrow \tilde{S}(Y) \text{ as,}$$

for irred. comp., we have field extension $k(Y_i)/k(X_i)$

we set $\tilde{S}(X_i) \longrightarrow \tilde{S}(Y_i)$ and

$$\tilde{S}(X) \longrightarrow \prod_{i \in X^{(0)}} \tilde{S}(X_i) \longrightarrow \prod_{i \in Y^{(0)}} \tilde{S}(Y_i) \longrightarrow \tilde{S}(Y).$$

- If f is a closed immersion, let N_f denote the normal bundle of f. We can consider the diagram

$$\begin{tikzcd} \mathbb{P}(N_f) \arrow[r, "\iota"] \arrow[d, "\pi'"'] & Bl_Z(X) \arrow[d, "\pi"] \\ Y \arrow[r, hook, "f"'] & X \end{tikzcd}$$

Note that π' is Zar. locally trivial map w/ projective space fibers. So, by construction, $\widetilde{S}$ only depends on function fields and since S is $\mathbb{A}^1$-invariant we have $\widetilde{S}(\mathbb{P}(N_f)) = \widetilde{S}(Y)$.
Since π is birat. and proper, again because $\widetilde{S}$ only depends on function fields, we have $\widetilde{S}(Bl_Z(X)) = \widetilde{S}(X)$. The morphism ι is a closed imm. of codim. 1 and there is a discrete val. arising from the blow-up and we use the residue extra-data to get a map

$$\begin{tikzcd} \widetilde{S}(X) = \widetilde{S}(Bl_Z X) \arrow[d] \\ \widetilde{S}(Y) = \widetilde{S}(\mathbb{P}(N_f)) \end{tikzcd}$$

- If f is a general morphism we can factorize it as follows:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\Gamma_f"'] & & X \\ & Y \times_k X \arrow[ur, "\pi"'] & \end{tikzcd}$$

and since Γ_f is a closed immersion and π is dominant we can define $\widetilde{S}(f)$ as

$\widetilde{S}(\Gamma_f) \circ \widetilde{S}(\pi)$.

By construction $\widetilde{S}$ is birat. and $\mathbb{A}^1$-invariant presheaf so, thanks to a lemma from the previous talk, $\widetilde{S}$ is a Nisnevich sheaf.
Last thing to check is compatibility and functoriality, this is precisely the following lemma:

<u>Lemma</u> (6.1.8)

- Given a sq. $\begin{tikzcd} Y' \arrow[r, hook] \arrow[d, "\text{smooth}"'] & Y \arrow[d, "\text{smooth}"] \\ X' \arrow[r, hook] & X \end{tikzcd}$

  the diagram $\begin{tikzcd} \widetilde{S}(X) \arrow[r] \arrow[d] & \widetilde{S}(X') \arrow[d] \\ \widetilde{S}(Y) \arrow[r] & \widetilde{S}(Y') \end{tikzcd}$ commutes.

- Given a sequence $Z \hookrightarrow Y \hookrightarrow X$ the triangle $\widetilde{S}(X) \to \widetilde{S}(Y) \to \widetilde{S}(Z)$ commutes.

Let (L, v) s.t. $L \in \widetilde{\mathcal{F}}_k$ and k_v/k sep., if X is a <u>proper</u> of f.t. over k, we can consider the $\widetilde{\mathcal{F}}_k$-set defined by

$L \longmapsto X(L)/\sim$ $\quad x_0, x_1 \in X(L)$

$x_0 \sim x_1$

$\iff$

$\exists\ \mathbb{A}^1 \xrightarrow{f} X$ s.t.

$f(0) = x_0,\ f(1) = x_1$.

and, thanks to the val. criterion of properness, the L-pts of X uniquely extend to $\mathcal{O}_v$-pts of X, i.e. we know that, the map $X(\mathcal{O}_v) \longrightarrow X(L)$ is a bijection.

$$\left(\begin{tikzcd} \mathrm{Spec}(L) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_v) \arrow[r] \arrow[ur, dashed, "\exists !"] & \mathrm{Spec}(k) \end{tikzcd}\right)$$

<u>Prop.</u> 0

If X is a proper scheme of f.t. over k, then,

<u>1</u>. There is a factorization

$$\begin{tikzcd} X(L) = X(\mathcal{O}_v) \arrow[r] \arrow[dr] & X(K_v) \arrow[r] & X(K_v)/\sim \\ & X(L)/\sim \arrow[ur, dashed, "\exists \rho_v"'] & \end{tikzcd}$$

hence we get a $\widetilde{\mathcal{F}}_k^1$-Set via the association

$$L \mapsto X(L)/\sim$$

2. The object obtained by the previous pt, denoted $\overline{\omega}_0^{b\mathbb{A}^1}(X)$, is sheaflike and $\mathbb{A}^1$-invariant.

In order to prove the prop. 0 we use the following results:

<u>Lemma</u> 1 Let S be an irred. ess. smooth k-scheme of dim. 2 w/ function field F, there $\exists$ finitely many closed pts $z_1, \dots, z_n \in S$,

Thanks to prop. 0 we can state the following result:

<u>Theorem</u> Suppose X is a proper scheme of f.t. over k.
There exists a birat. and $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$ together w/ a morphism $X \longrightarrow \pi_0^{b\mathbb{A}^1}(X)$ s.t.
$\forall L \in \mathcal{F}_k$, there is a factorization

$$\begin{tikzcd} X(L) \arrow[r] \arrow[dr] & \pi_0^{b\mathbb{A}^1}(X)(L) \\ & X(L)/\sim \arrow[u, dashed, "\simeq"'] \end{tikzcd}$$

<u>Pf.</u> Using prop. 0, we define $\pi_0^{b\mathbb{A}^1}(X)$ as the object of $Shv_k^{b\mathbb{A}^1}$ associated to $\overline{\omega}_0^{b\mathbb{A}^1}(X)$ via the equiv. of cat. given by the restriction functor.
It remains to construct the morphism

$$X \longrightarrow \pi_0^{b\mathbb{A}^1}(X),$$

we do it at the level of presheaves:
if $U \in Sm_k$ we can look at $X(U)/\sim$ the homotopy relation (i.e. $U \underset{g}{\overset{f}{\rightrightarrows}} X$ are homotopic $\Longleftrightarrow$

$$\begin{tikzcd} U \arrow[d] \arrow[dr, "f"] & \\ U \times \mathbb{A}^1 \arrow[r, "\exists"] & X \\ U \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$
)

and since this association is functorial there is a well-defined map $X(U) \longrightarrow X(U)/\sim \longrightarrow \pi_0^{b\mathbb{A}^1}(X)(U) \overset{\text{def.}}{=} X(k(U))/\sim$ induced by

$$\begin{tikzcd} \mathrm{Spec}(k(U)) \arrow[r] \arrow[dr] & U \arrow[d] \\ & \mathrm{Spec}(k) \end{tikzcd}.$$

In part., if $U = \mathrm{Spec}(L)$ we get the bijection

Finally,

<u>Prop.</u> Same hypothesis on X, there is a canonical map

- $\pi_0^{\mathbb{A}^1}(X) \longrightarrow \pi_0^{b\mathbb{A}^1}(X)$ s.t.
- $$\begin{tikzcd} \pi_0^{ch}(X) \arrow[r] \arrow[dr] & \pi_0^{b\mathbb{A}^1}(X) \\ & \pi_0^{\mathbb{A}^1}(X) \arrow[u] \end{tikzcd}$$ induces a bijection $\pi_0^{ch}(X)(L) \simeq \pi_0^{b\mathbb{A}^1}(X)(L)$, $\forall L \in \mathcal{F}_k$.

Pf. Since $\pi_0^{b\mathbb{A}^1}(X)$ is birat. and $\mathbb{A}^1$-inv. (by construction) it implies that $\pi_0^{b\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-local (Morel – Voevodsky char. of $\mathbb{A}^1$-loc.-objects)

Hence the univ. prop. of the $\mathbb{A}^1$-localization implies that the previous morphism

$$X \longrightarrow \pi_0^{b\mathbb{A}^1}(X)$$

factors uniquely through the $\mathbb{A}^1$-loc.:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \pi_0^{b\mathbb{A}^1}(X) \\ & L_{\mathbb{A}^1}(X) \arrow[u] \end{tikzcd}$$

Then, we have, $\forall U \in Sm$, a morphism

$$[U, L_{\mathbb{A}^1}(X)]_S \longrightarrow [U, \pi_0^{b\mathbb{A}^1}(X)]_S$$

which is functorial in U. But, since $\pi_0^{b\mathbb{A}^1}(X)$ is of simp. dim. 0 we have

$$[U, \pi_0^{b\mathbb{A}^1}(X)]_S = \pi_0^{b\mathbb{A}^1}(X)(U).$$

Hence, by def. of $\pi_0^{\mathbb{A}^1}(X)$, applying sheafification we obtain a morphism $\pi_0^{\mathbb{A}^1}(X) \longrightarrow \pi_0^{b\mathbb{A}^1}(X)$ from the previous morphism

Finally, the morphism $X \to L_{\mathbb{A}^1}(X)$ induces the morphism $X \longrightarrow \pi_0^{\mathbb{A}^1}(X)$ which factorises the morphism

$$\begin{tikzcd} X \arrow[r] \arrow[dr, two heads] & \pi_0^{\mathbb{A}^1}(X) \\ & \pi_0^{ch}(X) \arrow[u] \end{tikzcd}$$

In particular, we'll get a surjection between the k-pts of $\pi_0^{ch}(X)$ and $\pi_0^{\mathbb{A}^1}(X)$, i.e. we proved the thm 2.4.3.