$\mathbb{A}^1$-fundamental gp and $\mathbb{A}^1$-minimality

Today: end of the characterization of the $\mathbb{A}^1$-htpy type of surfaces via the $\mathbb{A}^1$-fdmntl gp and I'll explain some extensions

Next week: back to the following statement:

Thm If X is a proper scheme of f.t. over a field k the canonical epimorphism
$$\pi_0^{ch}(X) \twoheadrightarrow \pi_0^{\mathbb{A}^1}(X)$$
induces for every f.g.-separable ext. L/k a bijection $\pi_0^{ch}(X)(L) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(X)(L)$

Proposition Let X be a rat., smooth, proper surface over k alg. closed and let $x \in X(k)$. Then we have a $\mathbb{A}^1$-weak-eq.
$$X - x \underset{\mathbb{A}^1}{\simeq} (\mathbb{P}^1)^{\vee n}$$
with $n := \mathrm{rk}(\mathrm{Pic}(X))$

Rk. Since $\mathbb{P}^1$ play the rôle of the circle in motivic htpy theory, this fact is analogous to the fact in "classical" differential geom. that a closed 2-mfd has the htpy type of a wedge of circles.

Pf Recalling the $\mathbb{A}^1$-htpy classification of surfaces:

Thm Over an alg. closed field k the only (up to $\mathbb{A}^1$ w-eq.) k-rat. smooth, proper surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the S_i's where $S_i := \mathrm{Bl}_{I_i} \mathbb{P}^2$ with $i \geq 1$ and I_i is a finite set of distinct k-pts of $\mathbb{P}^2$.

- If $X = \mathbb{P}^2$ then choosing $x := (1:0:0) \in \mathbb{P}^2$ we get
$$\begin{aligned} \mathbb{P}^2 \setminus \{x\} &= \mathrm{Tot}_{\mathcal{O}_{\mathbb{P}^1}}(1) \\ &= \mathbb{A}^1 \times \mathbb{P}^1 \\ &\underset{\mathbb{A}^1}{\simeq} \mathbb{P}^1. \end{aligned}$$

- If $X = \mathbb{P}^1 \times \mathbb{P}^1$, it was already explained by Samuel in a more gen situation:

Consider $\{(\infty, \infty)\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and
$$\begin{aligned} Y := \mathbb{P}^1 \times \mathbb{P}^1 - \{(\infty,\infty)\} &= (\mathbb{P}^1 \times \mathbb{A}^1) \cup (\mathbb{A}^1 \times \mathbb{P}^1) \end{aligned}$$
and we have $(\mathbb{P}^1 \times \mathbb{A}^1) \cap (\mathbb{A}^1 \times \mathbb{P}^1) = \mathbb{A}^2$

So we obtain a diagram

$$\begin{tikzcd} & \mathbb{A}^2 \arrow[dl, hook'] \arrow[dr, hook] \arrow[dd, "\simeq"] & \\ \mathbb{P}^1 \times \mathbb{A}^1 \arrow[dd, "\simeq"'] & & \mathbb{A}^1 \times \mathbb{P}^1 \arrow[dd, "\simeq"] \\ & * \arrow[dl, hook'] \arrow[dr, hook] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

hence we have $Y \underset{\mathbb{A}^1}{\simeq} \mathbb{P}^1 \vee \mathbb{P}^1$.

For the general case we'll use the following lemma:

<u>Lemma</u> $Bl_0(\mathbb{A}^2)$ is homotpy equiv. to $\mathbb{A}^2 \setminus 0 \vee \mathbb{P}^1$

Pf. We start with $\overset{\simeq \mathbb{A}^2}{O_1}$ and $\overset{\simeq \mathbb{A}^2}{O_2}$ two open sets s.t. $Bl_0(\mathbb{A}^2) = O_1 \cup O_2$ and $O_1 \cap O_2 = \mathbb{G}_m \times \mathbb{A}^1$.
Assuming that the pt lie in a single copy of $\mathbb{A}^2$ we have a MV sq.

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 \setminus 0 \\ \downarrow & & \downarrow \\ \mathbb{A}^2 & \longrightarrow & Bl_0(\mathbb{A}^2) \end{array}$$

Since $\left| \begin{array}{l} \mathbb{G}_m \times \mathbb{A}^1 \underset{\mathbb{A}^1}{\simeq} \mathbb{G}_m \\ \mathbb{A}^2 \underset{\mathbb{A}^1}{\simeq} * \end{array} \right.$

we can replace the previous diagram by

$$* \longleftarrow \mathbb{G}_m \longrightarrow \mathbb{A}^2 \setminus 0$$

and $\mathbb{P}^1 = \operatorname{hocolim}(* \leftarrow \mathbb{G}_m \to C(\mathbb{G}_m))$, $C(\mathbb{G}_m) = \mathbb{G}_m \wedge \Delta^1_s$

we get $Bl_0(\mathbb{A}^2) \simeq \mathbb{P}^1 \vee \mathbb{A}^2 \setminus 0$.

- Let X be as in the prop. and $x \in X(k)$, we can choose an open subscheme of X isom. to $\mathbb{A}^2$ s.t. $x \in \mathbb{A}^2$ and consider $Y := Bl_x X$, writing D for the corresponding exc. divisor and choosing $y \in D$ we'll show that $Y \setminus y \underset{\mathbb{A}^1}{\simeq} X \setminus x \vee \mathbb{P}^1$.

Then the prop. will follows by induction.
Consider the MV square

$$\begin{array}{ccc} \mathbb{A}^2 \setminus 0 & \longrightarrow & Y \setminus D \\ \downarrow & & \downarrow \\ Bl_0(\mathbb{A}^2) \setminus y & \longrightarrow & Y \setminus y \end{array}$$

Since, by def., we have $Y \setminus D \simeq X \setminus x$ and, thanks to the lemma, we also have $Bl_0 \mathbb{A}^2 \setminus y \underset{\mathbb{A}^1}{\simeq} \mathbb{A}^2 \setminus 0 \vee \mathbb{P}^1$.

Because the map $\begin{array}{c} \mathbb{A}^2 \setminus 0 \\ \downarrow \\ Bl_0(\mathbb{A}^2) \setminus y \simeq \mathbb{A}^2 \setminus 0 \vee \mathbb{P}^1 \end{array}$ identifies

$\mathbb{A}^2 \setminus 0$ to the associated summand we get a cofib. $\mathbb{A}^2 \setminus 0 \hookrightarrow Bl_0 \mathbb{A}^2 \setminus y$ and

$$Y \setminus y \underset{\text{htpy}}{\simeq} \operatorname{hocolim}(\mathbb{P}^1 \leftarrow * \longrightarrow X \setminus x) \underset{\mathbb{A}^1}{\overset{\text{eq.}}{\simeq}} \mathbb{P}^1 \vee X \setminus x.$$

Cor. Under the same assumptions on X, for any $x \in X(k)$, we have

$$\underbrace{F_{\mathbb{A}^1}(1) *^{\mathbb{A}^1} \cdots *^{\mathbb{A}^1} F_{\mathbb{A}^1}(1)}_{m := rk(Pic(X)) \text{ times}} *_{F_{\mathbb{A}^1}(2)} e \xrightarrow{\simeq} \pi_1^{\mathbb{A}^1}(X)$$

Indeed, we've just seen that $\pi_1^{\mathbb{A}^1}(X \setminus x) \simeq \pi_1^{\mathbb{A}^1}(\mathbb{P}^{2 \vee m})$

and $\pi_1^{\mathbb{A}^1}(\mathbb{P}^{2 \vee m}) \simeq F_{\mathbb{A}^1}(1) *^{\mathbb{A}^1} \cdots *^{\mathbb{A}^1} F_{\mathbb{A}^1}(1)$

explained by Samuel last time

and we know that $\pi_1^{\mathbb{A}^1}(X \setminus x) *^{\mathbb{A}^1}_{F_{\mathbb{A}^1}(2)} e \simeq \pi_1^{\mathbb{A}^1}(X)$.

A priori, compared to the fact that

$$\pi_1^{\mathbb{A}^1}(Bl_{x_1, \dots, x_m}(\mathbb{P}^m)) \simeq \mathbb{G}_m *^{\mathbb{A}^1} \cdots *^{\mathbb{A}^1} \mathbb{G}_m$$

we don't have a better description in the gen. case but we can note the following fact:

In his book, F. Morel constructed a functor

$$C_*^{\mathbb{A}^1} : Sm(k) \longrightarrow D_{\mathbb{A}^1}(Ab(k))$$

and one can consider the first homology sheaf $H_1^{\mathbb{A}^1}(X)$ with $X \in Sm(k)$.

More precisely, if X is a surface and $x \in X(k)$, choosing $x \in \mathbb{A}^2 \subset X$ we can look at the MV sq.

$$\begin{array}{ccc} \mathbb{A}^2 \setminus 0 & \longrightarrow & X \setminus x \\ \downarrow & & \downarrow \\ \mathbb{A}^2 & \longrightarrow & X \end{array}$$

and we can show that $H_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \longrightarrow H_1^{\mathbb{A}^1}(X \setminus x)$ is an iso. via a MV sequence argument.

There is an $\mathbb{A}^1$-Hurewicz thm and since $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0)$ is abelian we get a factorization

$$\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \longrightarrow \pi_1^{\mathbb{A}^1}(X \setminus x)$$

through $\pi_1^{\mathbb{A}^1}(X \setminus x)^{ab}$, i.e. in a way $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \simeq \underline{K}_2^{MW}$ is in the commutator subgp of $\pi_1^{\mathbb{A}^1}(X, x)$.

Finally, we can state a new version of the $\mathbb{A}^1$-classification:

Thm If X and Y are rat., smooth, proper surfaces over an alg. closed field k, the following conditions are equiv.:

1. The varieties X and Y are $\mathbb{A}^1$-h-cobordant.
2. ——— $\mathbb{A}^1$-weak-equiv.
3. ——— have isom. $\mathbb{A}^1$-fund. gps.
4. The abelian gps $H^{2,1}(X, \mathbb{Z})$ and $H^{2,1}(Y, \mathbb{Z})$ equipped with the structure of quad.

4. $H^{2,1}(X,\mathbb{Z}) \simeq H^{2,1}(Y,\mathbb{Z})$
as quad. spaces.

Furthermore, the set $S_{\mathbb{A}^1}(X)$ consists of exactly 1 element.

$S_{\mathbb{A}^1}(X)$: consider the scheme structures on X, i.e. the pairs

(Y,s) with Y smooth, proper and
$s: Y \xrightarrow[\mathbb{A}^1]{\simeq} X$ and the equiv. rel.
gen. by $(Y,s) \sim (Y',s')$
$\iff$
$\exists$ a triple (W,f,H) with
(W,f) is an $\mathbb{A}^1$-h-cobordism
between Y and Y' and
$H: W \to X$ s.t.

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[rr, bend left, "s"] & W \arrow[r, "H"] & X \\ Y' \arrow[r, hook] \arrow[rr, bend right, "s'"'] & W \arrow[r, "H"] & X \end{tikzcd}$$

Precision on 4.:

Using the following result:

Thm (Voevodsky)
The motivic cohom. ring $\bigoplus_i H^{2i,i}(X,\mathbb{Z})$ and the Chow ring $CH^i(X)$ are isom.

and the fact that the motivic cohom. ring is an $\mathbb{A}^1$-invariant (due to Voevodsky)
and since the following
Lemma Two Hirzebruch surfaces $\mathbb{F}_a$ and $\mathbb{F}_b$ are $\mathbb{A}^1$-weakly-equiv. iff $a = b \ (2)$
is a consequence of the
Prop. The varieties $\mathbb{F}_{\underline{a}}$ and $\mathbb{F}_{\underline{a}'}$ are $\mathbb{A}^1$-w-eq. iff $\sum_i a_i = \sum_i a_i' \ [n+1]$
which is also a consequence of the fact that the associated Chow rings are isomorphic we finally have

Prop. Two rat., smooth, proper surfaces are $\mathbb{A}^1$-w.eq. iff the quad. forms on $H^{2,1}(X,\mathbb{Z})$ and $H^{2,1}(Y,\mathbb{Z})$ are isomorphic.

$\mathbb{A}^1$-htpy minimality:

Def Given an $\mathbb{A}^1$-htpy type represented by a smooth, proper k-var X we say that this htpy type is <u>minimal</u> if, for any triple (X', ψ, φ) with X' smooth, proper k-var.

$$\psi : X' \xrightarrow[\mathbb{A}^1]{\simeq} X$$

and $\varphi : X' \dashrightarrow Y$ proper, birat.

if φ is an iso.

- If X and Y are two smooth, proper k-var.
  X is an <u>$\mathbb{A}^1$-minimal model for Y</u>
  if: i). X is $\mathbb{A}^1$-minimal
  ii). $\exists X'$ smooth, proper k-var.
  s.t.

$$\begin{tikzcd} X' \arrow[r, "\simeq", "\mathbb{A}^1"'] \arrow[d, dashed, "\exists \text{ proper birat.}"] & Y \\ X & \end{tikzcd}$$

For example, the $\mathbb{A}^1$-minimal <u>rational</u> surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

Other ex. of $\mathbb{A}^1$-minimal varieties are given by <u>$\mathbb{A}^1$-rigid</u> varieties, i.e.

$\forall U \in Sm_k$, the map $X(U) \longrightarrow X(U \times \mathbb{A}^1)$ is a bijection.

Prop. The $\mathbb{A}^1$-htpy type of a smooth, proper $\mathbb{A}^1$-rigid k-var. X is $\mathbb{A}^1$-minimal.

Pf. Let $f : X \dashrightarrow Y$ proper birat.

We need the 1st lemma

<u>Lemma</u> (Abhyankar)

If Y is smooth of any dim. and $f : X \dashrightarrow Y$ proper birat. then, $\forall y \in Y$, either $f^{-1}(y)$ is a pt or $f^{-1}(y)$ is covered by rat. curves.

<u>Lemma</u>

Under the same assumptions, the exceptional set $Ex(f)$ is of pure codim 1.

Pf. (lemma 1)

- If Y is a surface then applying successively blowing-up at some pts of Y we get that, $\forall y \in Y$ $f^{-1}(y)$ is dominated by some $\mathbb{P}^1$'s. We can use the Lüroth's thm which alg. tells that if $k \subsetneq L \subseteq k(X)$ then $L \simeq k(X)$.

It follows from this result.

- In the more general case see the book of Mori and Kollár.

Now we come back to the pf of the initial prop.:

- If $Ex(f) = \emptyset$ by def., f is an isom.
- Otherwise, we can construct a non trivial morphism $\mathbb{A}^1 \longrightarrow X$ which is impossible because X is $\mathbb{A}^1$-rigid.

Back to connected and chain-connected components

Idea Construct a new sheaf $\pi_0^{b\mathbb{A}^1}(X)$ s.t. it possesses "special" prop. and we can factorize

$$\begin{tikzcd} \pi_0^{ch}(X) \arrow[r] \arrow[d] & \pi_0^{b\mathbb{A}^1}(X) \\ \pi_0^{\mathbb{A}^1}(X) \arrow[ur, "\exists"'] & \end{tikzcd}$$

Def. Let $S \in Psh(Sm_k)$, we say that S is birational if:

(1). $\forall X \in Sm_k$ with irred. components X_η ($\eta \in X^{(0)}$) the map
$$S(X) \longrightarrow \prod_{\eta \in X^{(0)}} S(X_\eta)$$
is a bijection.

(2). $\forall X \in Sm_k$, $\forall U \subseteq X$ open and dense the restriction $S(X) \longrightarrow S(U)$ is bijective.

Lemma Any birat. presheaf is automatically a Nisnevich sheaf.

We use the charact. of Nisnevich sheaves via distinguished squares.

Notation: $Shv_k^{b\mathbb{A}^1}$ is the full subcat. of Shv which are birat. and $\mathbb{A}^1$-invariant.

$\tilde{\mathcal{F}}_k$:= the cat. of f.g. separable ext. of k.

and $\tilde{\mathcal{F}}_k$-Set $:= [\tilde{\mathcal{F}}_k, Set]$.

Def. The cat. $\hat{\tilde{\mathcal{F}}}_k$-Set has the same objects than $\tilde{\mathcal{F}}_k$-Set but with additional data:

$\forall (L, v)$ with $L \in \tilde{\mathcal{F}}_k$ and v is a discrete val. on L with residue field K_v sep. on k there is a specialization morphism $s_v : S(L) \longrightarrow S(K_v)$.