The $\mathbb{A}^1$-fundamental grp of $\mathbb{P}^1$

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

$$\mathbb{P}^1 = \operatorname{hocolim}\left(\begin{array}{ccccc} * & \leftarrow & \mathbb{G}_m & \longrightarrow & C(\mathbb{G}_m) \\ & & & & \wr \\ & & & & \mathbb{G}_m \wedge \Delta^1_S \end{array} \right)$$

$$\longrightarrow \quad \mathbb{P}^1 \sim \Sigma^1_S \mathbb{G}_m .$$

<u>Recall</u> $\quad (S,s) \in Spc_{k,\bullet}$

$$F_{\mathbb{A}^1}(S,s) = \pi_1^{\mathbb{A}^1}(\Sigma^1_S (S,s))$$

free strongly $\mathbb{A}^1$-invariant sheaf of grps

$\mathcal{G}r_k^{\mathbb{A}^1}$ = cat. of strongly $\mathbb{A}^1$-invariant sheaves of grps

$\mathcal{G} \in \mathcal{G}r_k^{\mathbb{A}^1}$ $\qquad \mathrm{Hom}_{\mathcal{G}r_k^{\mathbb{A}^1}}(F_{\mathbb{A}^1}(S), \mathcal{G})$

$$\wr\!\wr$$

$$\mathrm{Hom}_{Spc_{k,\bullet}}(S, \mathcal{G})$$

$$(S, s) \longrightarrow F_{\mathbb{A}^1}(S, s)$$

<u>Def</u>: $F_{\mathbb{A}^1}(1) = F_{\mathbb{A}^1}(\underset{\substack{| \\ \text{pointed by } 1}}{\mathbb{G}_m}) = \pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$

$$\theta : \mathbb{G}_m \to F_{\mathbb{A}^1}(\mathbb{G}_m) = F_{\mathbb{A}^1}(1).$$

On the other hand, given

$f: \overline{X} \to X$ a $\mathbb{G}_m^{\times S}$-torsor

$\overline{X}, X$ smooth $\mathbb{A}^1$-connected, there is an exact sequence $1 \to \pi_1^{\mathbb{A}^1}(\overline{X}) \to \pi_1^{\mathbb{A}^1}(X) \to \mathbb{G}_m^{tr} \to 1.$

$\mathbb{A}^2 \setminus 0 \to \mathbb{P}^1$

$(x,y) \mapsto (x:y)$ $\quad \mathbb{G}_m$-torsor

$$\begin{tikzcd} \rightsquigarrow \quad 1 \arrow[r] & \pi_1^{\mathbb{A}^1}(\mathbb{A}^2\setminus 0) \arrow[r] & \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \arrow[r] \arrow[d, phantom, "\wr"] & \mathbb{G}_m \arrow[r] \arrow[l, bend right, "\theta"'] & 1 \\ & & F_{\mathbb{A}^1}(1) & & \end{tikzcd}$$

θ is claimed to be a set-theoretic splitting of $F_{\mathbb{A}^1}(1) \to \mathbb{G}_m$.

Prop: Let $m, n \in \mathbb{N}$ with $m > 0$
$n \geq 2$

$p_1, \dots, p_m \in \mathbb{A}^n_k$ k-rational

If $m > 1$, assume that k is infinite.

i) Then for any $q_1, \dots, q_m \in \mathbb{A}^1_k(k)$, there is an $\mathbb{A}^1$-weak equivalence

$$\Sigma_s^{n-1} \mathbb{G}_m^{\wedge(n-1)} \wedge (\mathbb{A}^1 \setminus \{q_1, \dots, q_m\}) \xrightarrow{\sim} \mathbb{A}^n \setminus \{p_1, \dots, p_m\}$$

ii) If $n > 2$ $\mathbb{A}^n \setminus \{p_1, \dots, p_m$ is $\mathbb{A}^1$-connected and

$$\pi_1^{\mathbb{A}^1}(\mathbb{A}^d \setminus \{p_1, \ldots, p_m\}) \xrightarrow{\sim} F_{\mathbb{A}^1}(\mathbb{G}_m^{\wedge(n-1)} \wedge (\mathbb{A}^1 \setminus \{q_1, \ldots, q_m\}))$$

Pf: ii) $\underline{n \geq 2}$ By the unstable $\mathbb{A}^1$-connectivity theorem,

$$\Sigma_S^{n-1} \mathbb{G}_m^{\wedge(n-1)} \wedge (\mathbb{A}^1 \setminus \{q_1, \ldots, q_m\})$$ is $\mathbb{A}^1$-$(n-2)$-connected

thus $\mathbb{A}^1$-1-connected.

For $n=2$ this is of the definition of $F_{\mathbb{A}^1}$

i) $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$

If k is infinite, then $\operatorname{Aut}(\mathbb{A}^n)$ acts m-transitively on $\mathbb{A}^n$: thus we may assume that

$$p_i = (q_i, 0, \ldots, 0)$$

If $m=1$, then $\operatorname{Aut}(\mathbb{A}^n)$ acts transitively on $\mathbb{A}^n$, so the assumption that k is infinite.

We now write

$$\mathbb{A}^n \setminus \{p_1, \dots, p_m\} = (\mathbb{A}^1 \setminus \{q_1, \dots, q_m\} \times \mathbb{A}^{n-1})$$

$$\cup \left(\mathbb{A}^1 \times \mathbb{A}^{n-1} \setminus 0 \right)$$

Thus $\mathbb{A}^n \setminus \{p_1, \dots, p_m\}$ is the homotopy push-out of

$$\mathbb{A}^1 \setminus \{q_1, \dots, q_m\} \leftarrow \mathbb{A}^1 \setminus \{q_1, \dots, q_m\} \times \mathbb{A}^{n-1} \setminus 0 \rightarrow \mathbb{A}^{n-1} \setminus 0 .$$

$$\rightsquigarrow \underset{\mathbb{A}^1}{\sim} \Sigma^1_s \, \mathbb{A}^1 \setminus \{q_1, \dots, q_m\} \wedge (\mathbb{A}^{n-1} \setminus 0).$$

→ induction on n

In particular,

$$\mathbb{A}^n \setminus 0 \sim \Sigma_s^{n-1} \, \mathbb{G}_m^{\wedge(n-1)} \wedge \overbrace{\mathbb{A}^1 \setminus 0}^{\mathbb{G}_m}$$

$$\sim \Sigma_s^{n-1} \wedge \mathbb{G}_m^{\wedge n}$$

For instance,

$$\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) = \begin{array}{c} F_{\mathbb{A}^1}(\mathbb{G}_m \wedge \mathbb{G}_m) \\ \| \\ F_{\mathbb{A}^1}(2) \end{array}$$

In fact, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) = \underline{K}_2^{MW}$.

Lemma: If G is a Nisnevich sheaf of gps then

$$G \to R\Omega_s^1 BG$$

is an simplicial weak equivalence. As a result $\pi_1^{\mathbb{A}^1}(G)$ is a Nisnevich sheaf of abelian gps.

. - - ?

$$SL_2 \longrightarrow \mathbb{A}^2 \setminus 0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,b)$$

is an $\mathbb{A}^1$-weak equivalence as it is Zariski-locally trivial with affine space fibres

In any case, $F_{\mathbb{A}^1}(2)$ is abelian

$$\begin{array}{c} F_{\mathbb{A}^1}(2) \\ \wr \\ \underline{K}_2^{MW}. \end{array}$$

We have an exact sequence

$$1 \to \underline{K}_2^{MW} \xrightarrow{i} F_{\mathbb{A}^1}(1) \overset{\theta}{\rightleftarrows} \mathbb{G}_m \to 1.$$

This is a *central* extension i.e. $\underline{K}_2^{MW} \to F_{\mathbb{A}^1}(1)$ identifies $\underline{K}_2^{MW}$ with the center of $F_{\mathbb{A}^1}(1)$. As a sheaf of *sets*,

$$F_{\mathbb{A}^1}(1) \simeq \underline{K}_2^{MW} \times \mathbb{G}_m$$

$$i(a)\,\theta(u) \mapsfrom (a, u)$$

There is a *symbol* morphism

$$\Phi : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m \wedge \mathbb{G}_m \to F_{\mathbb{A}^1}(\mathbb{G}_m \wedge \mathbb{G}_m) = \underline{K}_2^{MW}$$

$$(a, b) \longmapsto [a][b].$$

Up to an explicit auto. of $\underline{K}_2^{MW}$,

$$[a][b] = \Phi(a, b) = \theta(a)\,\theta(b)\,\theta(ab)^{-1}.$$

If $(a, a') \in (\underline{K}_2^{MW})^{\times 2}$, $(u, u') \in \mathbb{G}_m^{\times 2}$, then

$$a\,\theta(u)\;a'\,\theta(u') = a a\;\theta(u)\theta(u')$$

$$= a a'\,\Phi(u,u')\,\theta(uu').$$

The $\mathbb{A}^1$-fundamental gp of Hirzebruch surfaces

$a \in \mathbb{Z}$.

$$\mathbb{F}_a = \mathbb{P}\big(\mathcal{O} \oplus \underset{}{\overset{\mathbb{P}(\mathcal{O}(-a))}{\mathcal{O}(-a)}}\big) \longrightarrow \mathbb{P}^1.$$

The inclusion of a fibre, say over

$$\infty = [1:0]$$

yields a map $\mathbb{P}^1 \to \mathbb{F}_a$. We have a pull-back diagram

$$\begin{tikzcd}
\mathbb{F}_a \arrow[d] & \mathbb{A}^2\setminus 0 \times \mathbb{P}^1 \arrow[l] \arrow[d] \\
\mathbb{P}^1 & \mathbb{A}^2\setminus 0 \arrow[l, hook', "\mathbb{G}_m\text{-torsor}"']
\end{tikzcd}$$

hence we have a homotopy fibre sequence

$$\mathbb{P}^1 \to \mathbb{F}_a \overset{\curvearrowleft}{\to} \mathbb{P}^1.$$

This yields a split exact sequence

$$1 \to F_{\mathbb{A}^1}(1) \to \pi_1^{\mathbb{A}^1}(\mathbb{F}_a) \overset{\curvearrowleft}{\to} F_{\mathbb{A}^1}(1) \to 1 \quad (*)$$

Hence we have an iso

$$\pi_1^{\mathbb{A}^1}(\mathbb{F}_a) \xrightarrow{\sim} F_{\mathbb{A}^1}(1) \rtimes^a F_{\mathbb{A}^1}(1).$$

Rmk: i) $\pi_1^{\mathbb{A}^1}(\mathbb{F}_a) \xrightarrow{\sim} \begin{cases} F_{\mathbb{A}^1}(1) \times F_{\mathbb{A}^1}(1) & \text{if } a \in 2\mathbb{Z} \\ F_{\mathbb{A}^1}(1) \rtimes^1 F_{\mathbb{A}^1}(1) & \text{if } a \text{ is odd.} \end{cases}$

Furthermore, ii) $\pi_1^{\mathbb{A}^1}(\mathbb{F}_0) \not\simeq \pi_1^{\mathbb{A}^1}(\mathbb{F}_1)$.

(as sheaves of grps).

Pf: i) follows from the fact that

$$\mathbb{F}_a \simeq \mathbb{F}_b \iff a = b \mod 2$$

ii) $\mathbb{G}_m$ acts on $\mathbb{A}^2 \setminus 0$ by

$$v \cdot (x, y) = (v^0 x, v^a y)$$

Compatible with scaling; it yields

$$\begin{tikzcd}
\mathbb{G}_m \arrow[r] \arrow[dr] & \mathrm{Aut}(\mathbb{P}^1) \arrow[d] \\
 & \mathrm{Aut}(F_{/\mathbb{A}^1}(1))
\end{tikzcd}
\qquad , \quad v \cdot (x : y) = (x : v^a y) .$$

Claim: the action of $F_{/\mathbb{A}^1}(1)$ on itself given by $(*)$ factors through

$$\begin{tikzcd}
F_{/\mathbb{A}^1}(1) \arrow[r] \arrow[dr] & \mathbb{G}_m \arrow[d] \\
 & \mathrm{Aut}(F_{/\mathbb{A}^1}(1))
\end{tikzcd}$$

Assume the claim. Let L/k be finite type separable

$$u \in L^* \to F_{/\mathbb{A}^1}(1)(L) \to F_{/\mathbb{A}^1}(1)(L)$$

which is induced by

$$\mathbb{P}^1_L \to \mathbb{P}^1_L , \quad \begin{pmatrix} 1 & 0 \\ 0 & u^a \end{pmatrix} \in PGL_2(L)$$

For $a = 0$, this is the trivial map.

Assume $\underline{n=1}$. $\mathrm{Aut}(F_{\mathbb{A}^1}(1)) = \mathbb{Z} \oplus \underline{K}_1^{MW}$.

$$L^* \to \mathbb{Z} \oplus K_1^{MW}(L), \quad u \mapsto (1, [u]).$$

We want to understand the action of $(1,[u])$ on $F_{\mathbb{A}^1}(1)(L)$ by conjugation. This action is non-trivial (Morel's book).

$\mathbb{A}^1$-fundamental grps of surfaces: blow-ups and presentations.

If $n > 1$, $\pi^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0) = 1$. The exact sequence

$$1 \to \pi^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus 0) \to \pi^{\mathbb{A}^1}(\mathbb{P}^n) \to \mathbb{G}_m \to 1$$

shows that $\pi^{\mathbb{A}^1}(\mathbb{P}^n) = \mathbb{G}_m$.

X is <u>covered by affine spaces</u> if

$$X = \bigcup_{i \in I} U_i, \quad U_i \text{ open}, \quad U_i \simeq \mathbb{A}^n,$$

I finite, $U_i \cap U_j$ has a k-rational point for $i \neq j$.

Prop: Let X be covered by affine spaces, let $n = \dim X$ and let $x \in X(k)$

- If $n=2$, we have an iso

trivial sheaf of grps

$$\pi_1^{\mathbb{A}^1}(X \setminus x) \underset{\underline{K}_2^{MW}}{\overset{\mathbb{A}^1}{*}} e \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(X)$$

- If $n>2$, then

$$\pi_1^{\mathbb{A}^1}(X \setminus x) \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(X)$$

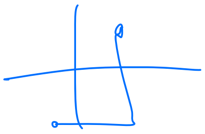
Furthermore we have isos

$$\pi_1(Bl_x(X)) \xrightarrow{\sim} \begin{cases} \underline{\pi_1^{\mathbb{A}^1}(X \setminus x) \underset{\underline{K}_2^{MW}}{\overset{\mathbb{A}^1}{*}} F_{\mathbb{A}^1}(1)} & \text{if } n=2 \\ \underline{\pi_1^{\mathbb{A}^1}(X) \overset{\mathbb{A}^1}{*} \mathbb{G}_m} & \text{if } n>2. \end{cases}$$

$$\operatorname{colim}(G \leftarrow H \rightarrow G) = G \underset{H}{\overset{\mathbb{A}^1}{*}} G' \quad \text{computed in } Gr_k^{\mathbb{A}^1}.$$

Strongly $\mathbb{A}^1$-invariant grps

dem : $\mathbb{A}^n\setminus 0$, $\mathbb{A}^n$ are $\mathbb{A}^1$-chain connected



$\rightsquigarrow$ $X\setminus x$ is covered by $\mathbb{A}^1$-chain connected opens and thus $\mathbb{A}^1$-connected.

$\ast$ if $X=\mathbb{A}^n$,

$$\pi^{\mathbb{A}^1}\left(\overset{\mathbb{A}^n\setminus 0}{\overline{X\setminus x}}\right)\underset{\underline{K}_2^{MW}}{\overset{\mathbb{A}^1}{\ast}} e = 1 \quad \text{if } n=2$$

$$\pi_1^{\mathbb{A}^1}(X\setminus x) = 1 \qquad \text{if } n\geq 3.$$

if $n=2$, $\pi_1^{\mathbb{A}^1}(X\setminus x)=\underline{K}_2^{MW}$

$$\rightsquigarrow \pi_1^{\mathbb{A}^1}(X\setminus x)\underset{\underline{K}_2^{MW}}{\overset{\mathbb{A}^1}{\ast}} e = 1$$

If $n\geq 3$, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^n\setminus 0)=1$

In general,

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] \\
X \setminus x \arrow[r] & X
\end{tikzcd}$$
is a push-out

with $\pi_1^{\mathbb{A}^1}(\mathbb{A}^n) = e$. The van Kampen theorem shows

$$\pi_1^{\mathbb{A}^1}(X \setminus x) \underset{\pi_1^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0)}{\ast} e \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(X).$$

If $n > 2$, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^n \setminus 0) = e$ and

$$\pi_1^{\mathbb{A}^1}(X \setminus x) \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(X).$$

If $\underline{n = 2}$, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) = \underline{K_2^{MW}}$ and this is the first statement.

* Blow-ups

$Bl_0(\mathbb{A}^n) =$ total space of $\mathcal{O}_{\mathbb{P}^{n-1}}(1) \to \mathbb{P}^{n-1}$

$\sim \mathbb{P}^{n-1}$

$\mathbb{A}^1$-weakly equivalent

$$\pi_1^{\mathbb{A}^1}(Bl_0(\mathbb{A}^n)) \simeq \begin{cases} F_{\mathbb{A}^1}(1) & \text{if } n=2 \\ \mathbb{G}_m & \text{if } n>2 \end{cases}$$

This solves the case $X \simeq \mathbb{A}^n$. In general, the diagram

$$\begin{tikzcd}
\mathbb{A}^n \setminus 0 \arrow[r] \arrow[d] & Bl_0(\mathbb{A}^n) \arrow[d] \\
X \setminus x \arrow[r] & Bl_x(X)
\end{tikzcd}$$

yields:

- if $n>2$, by van Kampen,

$$\underset{\pi_1^{\mathbb{A}^1}(X)}{\underset{\wr}{\pi_1^{\mathbb{A}^1}(X\setminus x)}} \overset{\mathbb{A}^1}{\underset{e}{\ast}} \underbrace{\pi_1(Bl_0(\mathbb{A}^n))}_{\substack{\wr \\ \mathbb{G}_m}} \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(Bl_x(X))$$

- if $n=2$, $\pi_1^{\mathbb{A}^1}(\mathbb{A}^2\setminus 0) = K_2^{MW}$

$$\pi^{\mathbb{A}^1}(Bl_0(\mathbb{A}^2)) = F_{\mathbb{A}^1}(1)$$

$$\rightarrow \pi^{\mathbb{A}^1}(X\setminus x) \underset{K_2^{MW}}{\ast} F_{\mathbb{A}^1}(1) \xrightarrow{\sim} \pi^{\mathbb{A}^1}(Bl_x(X)) \qquad \square$$

Corollary $\underline{m \geq 3}$

Then given $x_1, \ldots, x_n \in \mathbb{P}^m(k)$ distinct

$$\underbrace{\mathbb{G}_m \overset{\mathbb{A}^1}{*} \cdots \overset{\mathbb{A}^1}{*} \mathbb{G}_m}_{n \text{ copies}} \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}\left(Bl_{x_1, \ldots, x_n}(\mathbb{P}^m)\right)$$

We now turn to the case $\underline{m=2}$. We study

$$\pi_1^{\mathbb{A}^1}\left((\mathbb{P}^1)^{\vee n}\right)$$

In classical topology,

$$\pi_1(S^1 \vee \cdots \vee S^1) = \underset{\pi_1(S^1)}{\underset{\wr}{\mathbb{Z}}} * \cdots * \mathbb{Z}$$

Similarly:

Lemma: if $n \geq 1$,

$$\underbrace{F_{\mathbb{A}^1}(1) \overset{\mathbb{A}^1}{*} \cdots \overset{\mathbb{A}^1}{*} F_{\mathbb{A}^1}(1)}_{n \text{ factors}} \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}\left((\mathbb{P}^1)^{\vee n}\right).$$

Pf: Geometric model of $(\mathbb{P}^1)^{\times n}$.

Fix $\infty \in \mathbb{P}^1$. For $i < j$, consider

$$F_{ij} \hookrightarrow (\mathbb{P}^1)^{\times n} \quad : \quad F_{ij} = \{\text{i-th, j-th points} = \infty\}$$

$$X_n = (\mathbb{P}^1)^{\times n} \setminus \left(\bigcup_{i<j} F_{ij}\right).$$

$$= \bigcup_{i=1}^{n} X_{n,i}, \qquad X_{n,i} = (\mathbb{P}^1)^{\times n} \setminus \left(\bigcup_{j \neq i} F_{ij}\right)$$

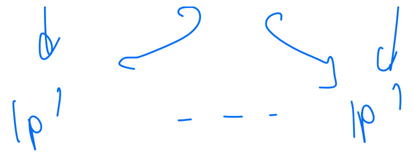
$X_{n,i} \hookrightarrow X_n$, $\quad X_{n,i} \simeq \mathbb{P}^1 \times \mathbb{A}^{n-1}$

$$X_{n,i} \cap X_{n,i'} = \mathbb{A}^n.$$

By a Van Kampen argument choosing $0 \in \mathbb{A}^n$ as a base point $(0 = ([0:1], \dots, [0:1]) \in (\mathbb{P}^1)^{\times n})$,

$$F_{\mathbb{A}^1}(1) \overset{\mathbb{A}^1}{*} \cdots \overset{\mathbb{A}^1}{*} F_{\mathbb{A}^1}(1) \xrightarrow{\sim} \pi_1^{\mathbb{A}^1}(X_n).$$

$$\begin{tikzcd}
 & & \mathbb{A}^n \arrow[dll, hook'] \arrow[d, "\wr"] \arrow[drr, hook] & & \\
\mathbb{P}^1 \times \mathbb{A}^{n-1} = X_{n,1} \arrow[d, "\wr"'] & \cdots & * & \cdots & X_{n,n} = \mathbb{P}^1 \times \mathbb{A}^{n-1} \arrow[d, "\wr"] \\
 & & & &
\end{tikzcd}$$



$$\rightsquigarrow \pi_1^{\mathbb{A}^1}((\mathbb{P}^1)^{\vee n}) \simeq \pi_1^{\mathbb{A}^1}(X_n) \simeq F_{\mathbb{A}^1}(1) *^{\mathbb{A}^1} \cdots *^{\mathbb{A}^1} F_{\mathbb{A}^1}(1).$$

Ex: let $a_i \in k$ distinct, $i = 1, \ldots, n+1$.

$$S = \{ xy = \prod_{i=1}^{n+1} (z - a_i) \} \subseteq \mathbb{A}^3$$

Let $\mathbb{A}^1_{n,0}$ be the non-separated smooth scheme obtained by gluing n copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus 0$ using Id. Then S is the total space of a flat morphism to $\mathbb{A}^1_{n+1,0}$ with fibres $\mathbb{A}^1$. This can be used to cover S by $(n+1)$-copies of $\mathbb{A}^2$ glued along $\mathbb{G}_m \times \mathbb{A}^1$, yielding

$$S \sim (\mathbb{P}^1)^{\vee n}.$$

Next time: M closed surface $x \in M$

$\to M \setminus x \sim S^1 \vee \cdots \vee S^1$

It will be shown that

<u>Prop</u>: Let X be a smooth proper rational surface $/k=\bar{k}$.
Let $n = \mathrm{rk}\,\mathrm{Pic}(X)$. If $x \in X(k)$, then

$$X \setminus x \sim (\mathbb{P}^1)^{\vee n}.$$