Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic h-cobordism: Near rationality and comparison results

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The letter k denotes a field. By *scheme*, we mean a separated k-scheme that is essentially of finite type as a k-scheme: by definition, this means that any scheme in our sense is a filtered limit of a diagram of finite type smooth k-schemes with smooth affine transition maps: the fibre product of schemes X and Y over $\operatorname{Spec} k$ is more simply denoted by $X \times Y$; similarly, if there is no further specification, $\mathbb{A}^n$ and $\mathbb{P}^n$ denote affine and projective n-space over k respectively. A *smooth k-scheme* is a scheme in this sense that is smooth over k and of finite type: the category of smooth k-schemes with all k-morphisms of schemes as morphisms is denoted by Sm_k. By *k-variety*, we mean an integral[1] scheme of finite type over k; the *function field* of a k-variety X is the stalk $k(X)$ of $\mathscr{O}_X$ at the generic point of X[2].

1. Birational geometry

First recall the following lemma about extending local morphisms.

Lemma 1.1. *Let* Y *and* X *be schemes, let x be a point of* X *and let y be a point of* Y*.*

- *Assume that* X *is of finite type at x. If f and f' are k-scheme morphisms from* Y *to* X *such that $f(y) = f'(y) = x$ and if the k-morphisms $\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ induced by f and f' are equal, then there exists an open neighbourhood* U *of y such that $f = f'$ on* U*.*

- *Assume that* X *is of finite presentation at x. Let $\varphi : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ be a local k-morphism. Then there exists opens $x \in U \subseteq X$ and $y \in V \subseteq Y$ and a scheme morphism $f : V \to U$ such that $\varphi = f^* : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$.*

- *Assume that* Y *is of finite presentation at y and that* X *is of finite presentation at x. Let $\varphi : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ be a k-isomorphism. Then there exist opens $x \in U \subseteq X$ and $y \in V \subseteq Y$ and an isomorphism $f : V \to U$ such that $\varphi = f^* : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$.*

Remark. Of course, since k is a field, finite type and finite presentation conditions coincide. The proof however will make it clear that X and Y could be schemes over any base (it is however then required to assume that x and y map to the same point of the base, since the statements do not make sense otherwise).

Proof. The question obviously being local in each case, we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are the spectra of k-algebras. Let $\mathfrak{p}$ (respectively $\mathfrak{q}$) be the prime ideal of A (respectively B) corresponding to x (respectively y).

[1] Hence non-empty.

[2] This stalk is a field: looking in any affine open neighbourhood $U = \operatorname{Spec} A$ of the generic point η, η corresponds to the zero ideal of A and $k(X)$ is simply the fraction field of A.

- Up to replacing X by an affine open neighbourhood of x whose ring of global sections is a finite-type k-algebra, we may assume that A is a finite-type k-algebra, say $A = k[t_1, \dots, t_n]/I$ for some ideal I. Let $\frac{t_i}{1}$ denote the image of t_i in $A_\mathfrak{p}$. Then by assumption, $f^*(\frac{t_i}{1}) = \frac{f^*t_i}{1}$ is equal to $f'(\frac{t_i}{1}) = \frac{f'^*t_i}{1}$ in B_q for all i, hence $s_i f^* t_i = s_i f'^* t_i$ for some $s_i \in B \setminus \mathfrak{q}$. Set $s = s_1 \cdots s_n$; then $\frac{f^*t_i}{1} = \frac{f'^*t_i}{1}$ in B_s for all i, thus the maps $A \to B_s$ are equal. Therefore $U = D(s)$ satisfies the condition of the claim.

- Again, we may assume that A is of finite presentation over k. Write $A = k[t_1, \dots, t_n]/\langle f_1, \dots, f_r \rangle$. As above, we see that there exists $s \in B \setminus \mathfrak{q}$ and, for each i, $b_i \in B$ such that $\varphi(\frac{t_i}{1}) = \frac{b_i}{s}$. We thus obtain a map $\psi : k[t_1, \dots, t_n] \to B_s$ by sending t_i to $\frac{b_i}{s}$ which sends f_i to 0, hence again as above there exists $s' \in B \setminus \mathfrak{p}$ such that $s'\psi(f_i) = 0$ for all i. Then ψ induces a map $A \to B_{ss'}$ which induces φ by localisation which produces the desired scheme map.

- There are affine open neighbourhoods U and U′ of x and affine neighbourhoods V and V′ of y as well as scheme morphisms $f : V \to U$ and $g : U' \to V$ such that $\varphi = f^* : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ and $\psi = g^* : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$. Since $f \circ g : U' \to U$ and the inclusion of U′ into U induce the same morphism $\mathscr{O}_{X,x}$, namely $\mathrm{Id}_{\mathscr{O}_{X,x}}$, the first item implies that there exists a neighbourhood U″ of x contained in U such that $f \circ g \circ j = j$ where $j : U'' \to U$ is the inclusion. Set $V'' = f^{-1}(U'')$. Then the map $g \circ j : U'' \to V$ factors through the open embedding $V'' \hookrightarrow V$: indeed, $f \circ g \circ j(x) = j(x) \in U''$ for all $x \in U''$ so that $g \circ j(x) \in V''$. Hence we now have morphisms $f : V'' \to U''$ and $g : U'' \to V''$.

  We replace notations with $X = U''$ and $Y = V''$ (we may, since U″ is an open neigbourhood of x and V″ is an open neighbourhood of y): we have constructed morphisms $f : Y \to X$ and $g : X \to Y$ such that $(f \circ g)^* : \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}$ and $(g \circ f)^* : \mathscr{O}_{Y,y} \to \mathscr{O}_{Y,y}$ are identity morphisms. If U is open in a scheme Z, we denote by j_U the corresponding open embedding. Then by the first item, $f \circ g \circ j_S = j_S$ for some open $S \subseteq X$. Setting $T = f^{-1}(S)$, $g \circ j_S$ factors through j_T as above so that we have morphisms $f : T \to S$ and $g : S \to T$ with $f \circ g = \mathrm{Id}_S$. Now again, $g \circ f \circ j_{T'} = j_{T'}$ for some open subset T′ of T. Setting $S' = g^{-1}(T')$, we obtain morphisms $g : S' \to T'$ and $f : T' \to S'$ such that $f \circ g = \mathrm{Id}_{S'}$ and $g \circ f = \mathrm{Id}_{T'}$ by construction.[3] This provides the desired isomorphism $f : T' \to S'$ inducing φ.

□

Corollary 1.2. *Let* X *and* Y *be k-varieties. Then the following assertions are equivalent.*

- *The function fields $k(X)$ and $k(Y)$ are isomorphic k-algebras.*
- *There exist non-empty opens $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $V \to U$ of k-varieties.*

Definition 1.3. If X and Y satisfy the equivalent assertions of the above corollary, we say that X and Y are *k-birational* or *k-birationally equivalent.*

The following notion is related though more restrictive.

Definition 1.4. Let $f : Y \to X$ be a morphism of k-varieties. Then f is a *birational morphism* if the induced map $f^* : k(X) \to k(Y)$ is an isomorphism.

Definition 1.5. Let X and Y be k-varieties. Let $f : U \to X$ and $g : V \to X$ be morphisms of k-schemes with $U \subseteq Y$ and $V \subseteq Y$ open and non-empty. We say that f and g are *equivalent* if $f \circ j_W = g \circ j_W$ for some non-empty open $W \subseteq U \cap V$ (here, j_W denotes the appropriate open embedding of W).

This defines an equivalence relation on k-scheme morphisms defined on non-empty open subsets of Y: a *rational map* φ from Y to X is an equivalence class for this relation and an element of this class is a *representative* of φ. Rational maps from Y to X are denoted using dashed arrows $\varphi : Y \dashrightarrow X$.

[3]The author acknowledges that the writing of this proof is poor but he did not manage to write the "ping-pong" of open subsets involved more clearly.

Let $\varphi : \mathrm{Y} \to \mathrm{X}$ be a rational map and let y denote the generic point of Y. Then $f(y) = g(y)$ if f and g are representatives of φ, hence $x = \varphi(y)$ is well-defined; so is the local morphism $\varphi^* : \mathscr{O}_{\mathrm{X},x} \to \mathscr{O}_{\mathrm{Y},y}$ induced by φ. In both cases, this is because any non-empty open of a k-variety contains its generic point.

The following lemma is a reformulation of Lemma 1.1.

Lemma 1.6. *Let* X *and* Y *be k-varieties; denote by y the generic point of* Y *and by* $\mathrm{R}(\mathrm{Y}, \mathrm{X})$ *the set of rational maps from* Y *to* X *and by* S *the set of pairs (x, f) where $x \in \mathrm{X}$ and f is a local morphism from $\mathscr{O}_{\mathrm{X},x}$ to $\mathscr{O}_{\mathrm{Y},y}$. Then the map $\varphi \mapsto (x, \varphi^* : \mathscr{O}_{\mathrm{X},\varphi(y)} \to \mathscr{O}_{\mathrm{Y},y})$ is a bijection from* $\mathrm{R}(\mathrm{Y}, \mathrm{X})$ *to* S.

Dominant rational maps.

Lemma 1.7. *Let* X *and* Y *be k-varieties and let $f : \mathrm{Y} \to \mathrm{X}$ be a morphism of k-schemes. Then the following assertions are equivalent.*

- *The image of the generic point of* Y *by f is the generic point of* X.
- *The set-theoretic image of f, that is, $f(\mathrm{Y}) = \{f(y), y \in \mathrm{Y}\}$, is dense in* X.

Proof. Let y be the generic point of Y. Then

$$\overline{f(\mathrm{Y})} = \overline{\{f(y)\}}.$$

Indeed, the inclusion $\overline{\{f(y)\}} \subseteq \overline{f(\mathrm{Y})}$ is obvious while, since f is continuous, $f(\mathrm{Y}) = f(\overline{\{y\}}) \subseteq \overline{\{f(y)\}}$ hence $\overline{f(\mathrm{Y})} \subseteq \overline{\{f(y)\}}$. The claim now follows from the fact that given an irreducible scheme X and any point $x \in \mathrm{X}$, the equality $\overline{\{x\}} = \mathrm{X}$ holds if, and only if, x is the generic point of X. □

Definition 1.8. A morphism of schemes $f : \mathrm{Y} \to \mathrm{X}$ is *dominant* if $f(\mathrm{Y})$ is dense in X. The above lemma shows that if Y and X are k-varieties, then f is dominant if, and only if, f sends the generic point of Y to the generic point of X.

A rational map $\varphi : \mathrm{Y} \dashrightarrow \mathrm{X}$ is dominant if φ has a dominant representative; if so, then any representative of φ is dominant.

Let $\varphi : \mathrm{Y} \to \mathrm{X}$ be a dominant rational map. Then by Lemma 1.6, φ comes from a unique k-algebra homomorphism $k(\mathrm{X}) \to k(\mathrm{Y})$. This correspondence should come from an equivalence of categories, hence the desire for a category of dominant rational maps.

Definition 1.9. Let $\varphi : \mathrm{Y} \dashrightarrow \mathrm{X}$ be a rational map. Let $y \in \mathrm{Y}$; then φ is *defined at y* if there exists a representative $f : \mathrm{U} \to \mathrm{X}$ of φ such that $y \in \mathrm{U}$. The *domain of definition* of φ is the set of points at which φ is defined: since it is given by

$$\bigcup_{(f:\mathrm{U}\to\mathrm{X})\in\varphi} \mathrm{U},$$

it is an open subset of Y.

Lemma 1.10. *Let $\varphi : \mathrm{Y} \dashrightarrow \mathrm{X}$ be a rational map with domain of definition* U. *Then φ has a unique representative $f : \mathrm{U} \to \mathrm{X}$.*

Proof. It suffices to show that any two representatives $f : \mathrm{V} \to \mathrm{X}$ and $f' : \mathrm{V}' \to \mathrm{X}$ agree on $\mathrm{V} \cap \mathrm{V}'$. Let $e : \mathrm{Z} \to \mathrm{V} \cap \mathrm{V}'$ be the equalizer of f and f' in the category of k-schemes. Then e is a closed immersion because X is separated by hypothesis ([Sta23, Tag 01KM]). On the other hand, Z contains an open subset of $\mathrm{V} \cap \mathrm{V}'$ by definition. Consequently, Z is equal to $\mathrm{V} \cap \mathrm{V}'$ topologically thus scheme-theoretically as Y is integral hence reduced. □

In general, one cannot compose rational maps: the problem is that the image of the first might not meet the domain of definition of the second in case the image of the first lies in a strict closed subset. Thus it should be no surprise that one can compose a *dominant* rational map with any rational map. Namely let $\psi : \mathrm{Z} \to \mathrm{Y}$ be a dominant rational map and let $\varphi : \mathrm{Y} \to \mathrm{X}$ be a rational map. Let $g : \mathrm{W} \to \mathrm{Y}$ be a representative of ψ and let $f : \mathrm{V} \to \mathrm{X}$ be a representative of φ. Then $g^{-1}(\mathrm{V}) = \mathrm{W}'$ is a non-empty open subset of W: indeed, since g is dominant, the generic point $z \in \mathrm{W}$ of Z is mapped to the generic point $y \in \mathrm{V}$. We define $\varphi \circ \psi$ as the rational map determined by $f \circ g_{|\mathrm{W}'}$. In particular, we obtain a category R_k whose objects are k-varieties and whose morphisms are dominant rational maps.

Proposition 1.11. *Let F_k be the category of field extensions of k of finite type. Then the functor $\mathsf{R}_k \to \mathsf{F}_k$ sending X to $k(\mathrm{X})$ and $\varphi : \mathrm{Y} \to \mathrm{X}$ to $\varphi^* : k(\mathrm{X}) \to k(\mathrm{Y})$ is an equivalence of categories.*

Proof. By now, it suffices to check that the functor is essentially surjective. If K/k is a field extension of finite type, then $\mathrm{K} = k(y_1, \ldots, y_n)$ for some elements $y_i \in \mathrm{K}$. Now $\mathrm{K} = k(\mathrm{X})$ where $\mathrm{X} = \operatorname{Spec} k[y_1, \ldots, y_n]$. □

2. Near-rationality

The following definition gives variation on the theme of *rationality*, that is, birationality to projective space.

Definition 2.1. Let X be a k-variety. Then X is called:

i) *k-rational* if X is k-birational to $\mathbb{P}^n$, in other words, if $k(\mathrm{X})$ is a purely transcendental extension of k or, equivalently, if some open subset of X is isomorphic to some open subset of $\mathbb{P}^n$ for some n;

ii) *stably k-rational* if there exists $n \geqslant 0$ such that $\mathrm{X} \times \mathbb{P}^n$[4] is k-rational;

iii) a *direct factor of a k-rational variety* or simply *factor k-rational* if there exists a k-variety Y such that $\mathrm{X} \times \mathrm{Y}$ is a k-rational variety;

iv) *retract k-rational* if there exists an open subscheme U of X such that there is a factorisation

$$\begin{array}{ccc} \mathrm{U} & \longrightarrow & \mathrm{U} \\ & \searrow \quad \nearrow & \\ & \mathrm{V} & \end{array}$$

where V is an open subscheme of the affine n-space $\mathbb{A}^n_k$, in other words, if X is a retract of $\mathbb{A}^n_k$ in R_k or equivalently of a k-rational variety;

v) *(separably) k-unirational* if $k(\mathrm{X})$ is a subfield of a transcendental extension of k (that is separable over $k(\mathrm{X})$) or equivalently if there exists a (separable) dominant rational map[5] from a projective space to X;

vi) *separably rationally connected* if there is a k-variety Y and a morphism $u : \mathrm{Y} \times \mathbb{P}^1 \to \mathrm{X}$ such that the map $u^{(2)} : \mathrm{U} \times_{\mathrm{Y}} \mathrm{U} \to \mathrm{X} \times \mathrm{X}$ is dominant and smooth at the generic point.[6]

We say that X is *rational* if $\mathrm{X} \times_k \operatorname{Spec} \overline{k}$ is $\overline{k}$-rational for an algebraic closure $\overline{k}$ of k.

Example 2.2. Any open subscheme of affine n-space is k-rational. In fact, X is k-rational if, and only if, X has a non-empty open subset isomorphic to an open subscheme of an affine space.

[4]This scheme is indeed a k-variety.

[5]The separability condition only depends on the induced field extension of rational functions and hence is indeed a property of rational maps (as opposed to morphisms).

[6]The scheme $\mathrm{U} \times_{\mathrm{Y}} \mathrm{U}$ is isomorphic to $\mathbb{P}^1 \times \mathrm{U}$ and is thus integral: indeed, $\mathbb{P}^1 \times \mathrm{Z}$ is integral if Z is an integral k-scheme which implies that $\mathrm{U} = \mathbb{P}^1 \times \mathrm{Y}$ is integral; hence, so is $\mathrm{U} \times \mathbb{P}^1$.

Lemma 2.3. *Let* X *be a* k*-variety. Then in the above definition, each of the first four conditions on* X *implies the next. If* X *is separably* k*-unirational, then* X *is separably rationally connected.*

The following proof is that of [CS06, Proposition 1.4]

Proof. Only the fact that iii) implies iv) merits a proof for the first assertion. Let then Y be a k-variety such that $\mathrm{X} \times \mathrm{Y}$ is a k-rational variety. Let U be a non-empty open subscheme of $\mathrm{X} \times \mathrm{Y}$ which is isomorphic to an open subscheme of an affine space. Let $(x_0, y_0) \in \mathrm{U}(k)$[7]. There exists a non-empty open subscheme X_1 such that $\mathrm{X}_1 \times \{y_0\} = \mathrm{U} \cap (\mathrm{X} \times \{y_0\})$. The set $\mathrm{U}_1 = \mathrm{U} \cap (\mathrm{X}_1 \times \mathrm{Y})$ is open in U and is thus isomorphic to an open subset of affine space. The map

$$\mathrm{X}_1 \xrightarrow{(x,y_0)} \mathrm{U}_1 \to \mathrm{X}_1$$

where the second map is induced by projection on X_1 is the expected retraction.

For the second claim, the authors of the article provide a reference to Kollár's book where the result is stated without proof. □

Example 2.4. We mention a few themes on near k-rationality.

- The Zariski or birational cancellation problem asked whether stably k-rational varieties are rational. This turns out to be false, even over $\mathbb{C}$ as shown by Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer. See [AM11, Example 2.3.4].

- If k is not algebraically closed, then there exist varieties that are factor k-rational yet not stably k-rational as shown by Colliot-Thélène and Sansuc. See [AM11, Example 2.3.5].

- Saltman introduced retract k-rationality in relation to Noether's problem regarding the rationality of fields of the form $k(\mathrm{V})^{\mathrm{G}}$ where G is a group acting on V.

- A notion of rational connectedness was introduced by Campana, Kollár, Miyaoka and Mori for fields of characteristic 0 but makes sense over any field. It is equivalent to separable rational connectedness if the base field is of characteristic 0 but is more well-behaved if the characteristic of k is positive. At the time of the publication of Asok and Morel's work, it was unknown whether separably rational connected varieties that are not unirational exist (the author does not know whether this has changed).

Near rationality and $\mathbb{A}^1$-connectedness

We now relate (weak) $\mathbb{A}^1$-connectedness with rationality properties of algebraic varieties. For k algebraically closed of characteristic 0, it was initially hoped that $\mathbb{A}^1$-connectedness would be equivalent to separable rational connectedness but we shall see obstructions for this to be true in this and future talks of different natures.

Proposition 2.5 ([CS77, Proposition 10])**.** *Let* k *be a perfect field. Let* Y *and* X *be* geometrically *irreducible*[8] *smooth proper* k*-varieties. Assume that either:*

1. *the morphism* f *is a blow-up of* X *at a smooth closed subscheme* Z *of codimension* $r+1$.

2. *the field* k *is of characteristic* 0 *and that* f *is any birational morphism.*

Then for any finitely generated separable field extension L/k*,* f *induces a map of* $\mathbb{A}^1$*-equivalence classes of* L*-points* $\mathrm{Y}(\mathrm{L})/\sim \to \mathrm{X}(\mathrm{L})/\sim$ *and this map is a bijection. Hence,* Y *is* $\mathbb{A}^1$*-chain connected if, and only if,* X *is* $\mathbb{A}^1$*-chain connected.*

[7] Such an (x_0, y_0) exists as soon as k is infinite. If k is finite, the complement of k^n in $\mathbb{A}^n_k$ is obviously an open subset without k-points. We do not know whether the assertion that iii) implies iv) holds if k is finite.

[8] We were unable to clarify what happens with $\mathbb{A}^1$-equivalence classes of L-rational points for field extensions L/k without this hypothesis.

Proof. The fact that f induces a map is quite general. Let us first assume that we are in the case 1. Note that since the assumptions are stable under base change, we can restrict our attention to the case where $\mathrm{L} = k$. We then follow the proof of [CS77]. To prove that the map $\mathrm{Y}(k)/\!\sim \to \mathrm{X}(k)/\!\sim$ induced by f is surjective, it suffices to show that the map $f(k) : \mathrm{Y}(k) \to \mathrm{X}(k)$ induced by f on k-rational points is surjective. But by assumption on f, for any $x \in \mathrm{X}(k)$, the morphism $f^{-1}(x) \to \operatorname{Spec} k$ is either an isomorphism (if x does not belong to Z) or, up to k-isomorphism, the structure morphism $\mathbb{P}^d_k \to \operatorname{Spec} k$. In any case, it is non-empty as claimed.

Let us now show that the map $\mathrm{Y}(k)/\!\sim \to \mathrm{X}(k)/\!\sim$ is *injective*. We first prove the following result:

Lemma 2.6. *Let $h : \mathbb{A}^1_k \to \mathrm{X}$ be an $\mathbb{A}^1$-elementary equivalence between k-points x and x'. Then there exists a sequence $(y_0, \dots, y_r)$ of k-rational points of* Y *and for each i, an elementary $\mathbb{A}^1$-equivalence between y_i and y_{i+1}, such that $f(y_0) = x$ and $f(y_r) = x'$.*

Proof. We have that there exist $y_0, \dots, y_r$ with $y_0 = y$ and $y_r = y'$, and an elementary $\mathbb{A}^1$-equivalence from y_i to y_{i+1} for all $i < r$.

First assume that the image of h is not contained in Z. Then there is a unique lift $\mathrm{H} : \mathbb{A}^1_k \to \mathrm{Y}$ of h along f.[9]. The point H(0) (respectively H(1)) lies over x (respectively y) as required.

Let us then assume that $h(\mathbb{A}^1_k)$ is contained in Z. First assume that x and x' both belong to some open U of Z such that $\mathrm{U} \times_{\mathrm{X}} \mathrm{Y}$ is k-isomorphic to $\mathrm{U} \times \mathbb{P}^r_k$. Fix $a \in \mathbb{P}^r(k)$; then the formula $t \mapsto (h(t), a)$ is well-defined on a non-empty open subset of $\mathbb{A}^1$, namely the inverse image of U by h, which contains 0 and 1 as x and x' belong to U, hence it extends to a map $\mathrm{H} : \mathbb{A}^1_k \to \mathrm{Y}$ by properness of Y. Then $f(\mathrm{H}(0)) = f((h(0), a)) = h(0) = x$ and $f(\mathrm{H}(1)) = f((h(1), a)) = h(1) = x'$ as required.

Now in general, by definition of blow-ups, Z may be covered by a finite number of open subsets with the same property as the open U above, which concludes the proof. □

Now let y and y' be k-points of Y such that $f(y) = x$ and $f(y') = x'$ are $\mathbb{A}^1$-equivalent. By assumption, there exists a sequence $(x_0, \dots, x_r)$ of k-points of X such that $x_0 = x$ and $x_r = x'$ and for each $i < r$, an elementary $\mathbb{A}^1$-equivalence between x_i and x_{i+1}. For each i, there exists a family $(y^i_0, \dots, y^i_{s_i})$ of k-points of Y such that $f(y^i_0) = x_i$ and $f(y^i_{s_i}) = x_{i+1}$ and an elementary $\mathbb{A}^1$-equivalence between y^i_j and y^i_{j+1}. Now as we have observed, the fibres of f over k-points are either a point or an r-dimension projective space: as a result, they are $\mathbb{A}^1$-chain connected so that there is an elementary $\mathbb{A}^1$-equivalence between $y^i_{s_i}$ and y^{i+1}_0 for all i, as well elementary $\mathbb{A}^1$-equivalences between y and y^0_0 and between $y^r_{s_r}$ and y'. Chaining these together, we see that $y \sim y'$ as required.

Now we treat the case 2, that is, we assume that f is birational and that k is of characteristic 0. By Hironaka's resolution of singularities [Hir69], there exists a diagram:

$$\begin{array}{ccc} \mathrm{Y}' & \longrightarrow & \mathrm{X}' \\ \downarrow & \swarrow & \downarrow \\ \mathrm{Y} & \longrightarrow & \mathrm{X} \end{array}$$

of birational morphisms, where the *vertical* maps are obtained as compositions of blow-ups as in Case 1. It then follows that in the induced commutative diagram of sets:

$$\begin{array}{ccc} \mathrm{Y}'(k)/\!\sim & \longrightarrow & \mathrm{X}'(k)/\!\sim \\ \downarrow & \swarrow & \downarrow \\ \mathrm{Y}(k)/\!\sim & \longrightarrow & \mathrm{X}(k)/\!\sim \end{array}$$

the vertical maps are bijections. Hence the diagonal map is a bijection (it is a surjection by the left triangle and an injection by the right triangle), thus the horizontal maps, in particular, the bottom one, is a bijection. □

[9]Indeed, the $k(t)$-point η of $\mathbb{A}^1$ is not sent to a point in Z, for otherwise, $\overline{h(\mathbb{A}^1)} = \overline{h(\eta)}$ would also be included in Z (which is closed): this is forbidden by assumption. This $k(t)$-point lifts to a point of Y because blow-ups are surjective on K-rational points for any K/k, hence a morphism $\mathrm{H} : \mathbb{A}^1_k \to \mathrm{Y}$ by properness of Y. The maps $f \circ \mathrm{H}$ and h agree at the generic point by assumption hence on a non-empty open subset of $\mathbb{A}^1_k$ thus on $\mathbb{A}^1_k$ since $\mathbb{A}^1_k$ is reduced and Y is separated.

Corollary 2.7. *Under the hypotheses of 1. in the previous proposition above,* X *is (weakly)* $\mathbb{A}^1$*-chain connected if, and only if,* Y *is (weakly)* $\mathbb{A}^1$*-chain connected.*

Definition 2.8. We say that *weak factorisation holds over* k *in dimension* n if given any two k-birationally equivalent smooth proper varieties X and X′ of dimension n, there exist smooth proper k-varieties $\mathrm{Z}_1, \dots, \mathrm{Z}_n, \mathrm{X}_1, \dots, \mathrm{X}_n$ of dimension n and a diagram

$$\mathrm{X} \leftarrow \mathrm{Z}_1 \rightarrow \mathrm{X}_1 \leftarrow \mathrm{Z}_2 \rightarrow \cdots \leftarrow \mathrm{Z}_{n-1} \rightarrow \mathrm{X}_n \leftarrow \mathrm{Z}_n \rightarrow \mathrm{X}'$$

with every morphism of with source Z_i a blow-up at a smooth center.

Example 2.9. If k is of characteristic 0, then weak factorisation holds over k in dimension n for all n by [Abr+02].

Theorem 2.10 ([AM11, Theorem 2.3.6])**.** *Let* k *be a perfect field and assume that weak factorisation holds over* k *in dimension* n*.*

i) If X *and* X′ *are smooth proper* k*-varieties of dimension* n *and if* X *and* X′ *are* k*-birationally equivalent, then* X *is (weakly)* $\mathbb{A}^1$*-chain connected if, and only if,* X′ *is (weakly)* $\mathbb{A}^1$*-chain connected.*

Assume further that k *is of characteristic* 0*.*

ii) If X *is a retract* k*-rational geometrically irreducible variety, then* X *is* $\mathbb{A}^1$*-chain connected, hence* $\mathbb{A}^1$*-connected.*

Before we proceed to the proof, we make an observation. Assume for a moment that k is of characteristic 0. Let $\varphi : \mathrm{Y} \dashrightarrow \mathrm{X}$ be a rational map between geometrically irreducible varieties. Let $f : \mathrm{V} \rightarrow \mathrm{X}$ be a representative of f with $\mathrm{V} \subseteq \mathrm{Y}$ open. By our assumptions on schemes, more specifically, because $\mathrm{X} \rightarrow \operatorname{Spec} k$ is separated, the graph morphism $\Gamma_f : \mathrm{V} \rightarrow \mathrm{V} \times \mathrm{X}$ is a closed immersion. Let Γ denote its image, which is closed *in* $\mathrm{V} \times \mathrm{X}$, and let Γ denote its (reduced) closure in $\mathrm{Y} \times \mathrm{X}$. Again, by Hironaka's work [Hir69], $\overline{\Gamma}$ admits a resolution of singularities, that is, there exists a smooth geometrically irreducible k-variety Y′ and a proper birational morphism $p : \mathrm{Y}' \rightarrow \overline{\Gamma}$ which may be viewed as a map $\pi : \mathrm{Y}' \rightarrow \mathrm{Y} \times \mathrm{X}$, yielding morphisms $\mathrm{Y}' \rightarrow \mathrm{Y}$ and $\mathrm{Y}' \rightarrow \mathrm{X}$ via the projections. Let us make a few comments.

- The projection map induces an isomorphism $\mathrm{pr} : \Gamma \rightarrow \mathrm{V}$. Consequently, if V is smooth, for instance if Y is smooth, then Γ is a smooth open subset of $\overline{\Gamma}$ and still according to Hironaka's result, there is an induced isomorphism $(\mathrm{pr} \circ p)^{-1}(\mathrm{V}) \rightarrow \mathrm{V}$. In any case, since p is birational, $\mathrm{pr} \circ p$ is also birational and so is the morphism $\mathrm{Y}' \rightarrow \mathrm{Y}$ as a result.

- Let W be an open subset of V such that the map $\mathrm{Y}' \rightarrow \mathrm{Y}$ is an isomorphism on W (W can be taken equal to V if V is smooth, for instance if Y is smooth). Then the map $\mathrm{W} \rightarrow \mathrm{Y}' \rightarrow \mathrm{X}$ is the restriction of f to W by definition. Hence the map $\mathrm{Y}' \rightarrow \mathrm{Y}$, which is birational, induces an isomorphism $\mathrm{Y} \dashrightarrow \mathrm{Y}'$ in R_k and the following diagrams

$$\begin{array}{ccc} \mathrm{Y} & \dashrightarrow & \mathrm{X} \\ \downarrow & \nearrow & \\ \mathrm{Y}' & & \end{array} \qquad \begin{array}{ccc} \mathrm{W} & \longrightarrow & \mathrm{X} \\ \downarrow & \nearrow & \\ \mathrm{Y}' & & \end{array}$$

  are commutative.

We believe this construction is what the authors refer to as *resolution of indeterminacy* in the proof of [AM11, Theorem 2.3.6]. In that case, the map $f : \mathrm{V} \rightarrow \mathrm{X}$ factors through an open subset of U of X and by assumption, there exists $g : \mathrm{U} \rightarrow \mathrm{V}$ such that $f \circ g = \mathrm{Id}_{\mathrm{U}}$. Then the above construction, applied to the rational map $\mathrm{X} \dashrightarrow \mathrm{Y} \dashrightarrow \mathrm{Y}'$ induced by $\mathrm{U} \rightarrow \mathrm{V} \dashrightarrow \mathrm{Y}'$, yields a smooth k-variety X′ and morphisms $\mathrm{X}' \rightarrow \mathrm{X}$ and $\mathrm{X}' \rightarrow \mathrm{Y}'$ such that the map $\mathrm{X}' \rightarrow \mathrm{X}$ is birational. We therefore get a commutative diagram:

$$\begin{array}{ccccc} X & \dashrightarrow & Y & \dashrightarrow & X \\ \uparrow & & \uparrow & \nearrow & \\ X' & \longrightarrow & Y' & & \end{array}$$

in R_k, where the vertical arrows are isomorphisms in R_k, that is, birational morphisms. In particular, if U is smooth (for instance if X is smooth), then $X' \to X$ is an isomorphism on U and the induced map $U \to X' \to Y' \to X$ is none other than $f \circ g$, namely Id_U. In particular, the morphism $X' \to Y' \to X$ is an isomorphism on U and is therefore birational.

Proof of Theorem 2.10. By the assumption that weak factorisation holds over k in dimension n, Claim i) reduces to Case 1. in Proposition 2.5.

Let us establish ii): henceforth, we assume that k is of characteristic 0. There exists an open subscheme $U \subseteq X$ and an open subscheme V of $\mathbb{A}^m$ such that $\mathrm{Id}_U : U \to U$ factors through V. As discussed above, there exists a commutative diagram

$$\begin{array}{ccccc} X & \dashrightarrow & \mathbb{A}^m & \dashrightarrow & X \\ \uparrow & & \uparrow & \nearrow & \\ X' & \longrightarrow & Y' & & \end{array}$$

whose solid arrows are scheme morphisms and whose vertical arrows are birational morphisms. Moreover, the morphisms is birational. Now let L be a finitely generated separable extension. Since $\mathbb{A}^m$ is $\mathbb{A}^1$-chain connected, $\mathbb{A}^m(L)/\!\sim\, = *$ so that $Y'(L)/\!\sim\, = *$ by Case 2. in Proposition 2.5 (which we may apply since the varieties considered are all geometrically irreducible). By the same case, the composite map

$$X'(L)/\!\sim\; \to Y(L)/\!\sim\; \to X(L)/\!\sim$$

is a bijection. In particular, the map $Y(L)/\!\sim\; \to X(L)/\!\sim$ is surjective hence $X(L)/\!\sim\, = *$ and X is $\mathbb{A}^1$-chain connected as required. □

Since weak factorisation holds over k in dimension 2, that is, for surfaces for any k perfect, the following also holds:

Corollary 2.11 ([AM11, Corollary 2.3.7])**.** *If k is perfect, then any k-rational smooth proper surface is $\mathbb{A}^1$-connected.*

Let now k^s be a separable closure of k. Let X be a k-variety and let $x \in X(k^s)$. Then X is *strongly rationally connected* relative to x if for any k^s-point y of X, x and y lie in the image of a morphism $\mathbb{P}^1 \to X$; X is strongly rationally connected if there is a dense open subset U of X such that X is strongly rationally connected relative to x for all $x \in U(k^s)$[10]. The theorem below [AM11, Theorem 2.3.9] is claimed to follow from the definitions, from the fact that weak $\mathbb{A}^1$-chain connectedness implies weak $\mathbb{A}^1$-connectedness and from the fact that strong rational connectedness for a smooth proper k-variety is invariant under separably closed extension (we tried to check the reference to [Kol96] given for this last fact but were unable to find it).

> Suppose that k is perfect. Then any strongly rationally connected smooth proper k-variety is weakly connected.

The following is proposed as a corollary [AM11, Corollary 2.3.10] but we were again unable to find an argument (we believe that the argument is that a strongly rationally connected variety is separably rationally connected but we do not know enough to ensure the accuracy of this belief).

> Suppose that k is perfect. Then any separably rationally connected smooth proper k-variety is weakly $\mathbb{A}^1$-connected.

[10] This reflects the author's understanding of the definition but is not the definition given in the reference [HT08] provided in [AM11]. In [HT08], one only asks that any point of X may be joined to the generic point of X by a rational curve $\mathbb{P}^1 \to X$ in x. The setting is slightly different, however, since we believe the general assumption in [HT08] is that k is algebraically closed of characteristic 0.

Finally, let us mention that at the time of publication of [AM11], it was unknown whether $\mathbb{A}^1$-chain connectedness is equivalent to retract k-rationality.

3. Comparison results

3.1 Comparison with étale $\mathbb{A}^1$-connectedness

In this section, we compare $\mathbb{A}^1$- and étale $\mathbb{A}^1$-connectedness (recall that the first notion refers to the Nisnevich topology).

Let $\alpha : (\mathsf{Sm}_k)_{\text{ét}} \to (\mathsf{Sm}_k)_{\text{Nis}}$ denote the morphism of sites induced by the identity of Sm_k: this is well-defined because the Nisnevich topology is weaker than the étale topology. The map α_* is the identity on sheaves (more precisely, it sends an étale sheaf to the underlying Nisnevich sheaf) and the map α^* is the étale sheafification functor. They induce functors

$$\alpha_* : \mathsf{Spc}_k^{\text{ét}} \to \mathsf{Spc}_k, \quad \alpha^* : \mathsf{Spc}_k \to \mathsf{Spc}_k^{\text{ét}}$$

between categories of simplicial sheaves. These functors form a Quillen adjunction which induces an adjunction

$$\alpha^* : \mathcal{H}^{\text{ét}}(k) \to \mathcal{H}^{\text{Nis}}(k), \quad \mathbf{R}\alpha_* : \mathcal{H}^{\text{Nis}}(k) \to \mathcal{H}^{\text{ét}}(k)$$

where $\mathbf{R}\alpha_*$ is the right derived functor of α_*.

Now let $\mathscr{X}$ be a simplicial étale sheaf on Sm_k. By adjunction, we have a bijection

$$\mathrm{Hom}_{\mathcal{H}^{\text{ét}}(k)}(\mathrm{U}, \mathscr{X}) \xrightarrow{\cong} [\mathrm{U}, \mathbf{R}\alpha_*\mathscr{X}]_{\mathbb{A}^1}$$

(indeed, since the étale topology is sub-canonical, $\alpha^*\mathrm{U} = \mathrm{U}$); here recall that

$$[\mathrm{U}, \mathbf{R}\alpha_*\mathscr{X}]_{\mathbb{A}^1} = \mathrm{Hom}_{\mathcal{H}(k)}(\mathrm{U}, \mathbf{R}\alpha_*\mathscr{X})$$

where $\mathcal{H}(k)$ is constructed using the Nisnevich topology. Now the unit map $\mathscr{X} \to \alpha^*\mathbf{R}\alpha_*\mathscr{X}$[11] of the adjunction provides a map

$$[\mathrm{U}, \mathscr{X}]_{\mathbb{A}^1} \to [\mathrm{U}, \mathbf{R}\alpha_*\mathscr{X}]_{\mathbb{A}^1} \to \mathrm{Hom}_{\mathcal{H}^{\text{ét}}(k)}(\mathrm{U}, \mathscr{X}).$$

Let $a_{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathscr{X})$ be the étale sheafification of the presheaf $\mathrm{U} \mapsto [\mathrm{U}, \mathscr{X}]_{\mathbb{A}^1}$. Then sheafification yields a morphism

$$a_{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathscr{X}) \to \pi_0^{\mathbb{A}^1,\text{ét}}(\mathscr{X})$$

of étale sheaves. The authors then make the following claim [AM11, Lemma 2.4.1].

Lemma 3.1 ([AM11, Lemma 2.4.1])**.** *The morphism $a_{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathscr{X}) \to \pi_0^{\mathbb{A}^1,\text{ét}}(\mathscr{X})$ is an epimorphism of étale sheaves. Hence if the simplicial Nisnevich sheaf underlying $\mathscr{X}$ is $\mathbb{A}^1$-connected, then $\mathscr{X}$ is étale $\mathbb{A}^1$-connected.*

Proof. This morphism factors as

$$a_{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathscr{X}) \to a_{\text{ét}}\pi_0^{\mathbb{A}^1}(\mathbf{R}\alpha_*\mathscr{X}) \to \pi_0^{\mathbb{A}^1,\text{ét}}(\mathscr{X})$$

where the second is a section-wise bijection at the level of presheaves hence it suffices to show that the first one is an epimorphism of étale sheaves. To do this, we prove that it is an epimorphism stalk-wise. Let $\mathrm{L}_{\mathbb{A}^1}$ denote the Nisnevich $\mathbb{A}^1$-localisation functor. Then we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{X} & \longrightarrow & \mathbf{R}\alpha_*\mathscr{X} \\ \downarrow & & \downarrow \\ \mathrm{L}_{\mathbb{A}^1}\mathscr{X} & \longrightarrow & \mathrm{L}_{\mathbb{A}^1}\mathbf{R}\alpha_*\mathscr{X} \end{array}$$

[11] This is thus a morphism in the homotopy category.

It suffices to show that the bottom horizontal map induces an epimorphism of sheaves of étale simplicial connected components. Since the vertical maps are epimorphisms at étale stalks by the étale $\mathbb{A}^1$-0-connectedness theorem, it suffices to show that this property holds for the top horizontal map. But $\mathbf{R}\alpha_* = \alpha_* \circ \mathrm{Ex}^{\text{ét}}$ where $\mathrm{Ex}^{\text{ét}}$ is the étale simplicial fibrant replacement functor so that the map $\mathscr{X} \to \mathbf{R}\alpha_*\mathscr{X}$ is a fibrant replacement functor for simplicial sets at étale stalks: as such, it is 0-connected which implies the desired property. □

Although we were not able to verify it, this lemma is claimed by the authors to imply that a variety that becomes $\mathbb{A}^1$-connected over a separable closure of k is étale $\mathbb{A}^1$-connected. The authors provide an example [AM11, Example 2.4.2] of an $\mathbb{R}$-variety which is claimed to be étale $\mathbb{A}^1$-connected because it is $\mathbb{A}^1$-connected after extension of scalars to $\mathbb{C}$ but is certainly not $\mathbb{A}^1$-connected as can be deduced from the disconnectedness of its real topological realisation.

3.2 $\mathbb{A}^1$-connectedness and $\mathbb{A}^1$-chain connectedness

Recall that given a space $\mathscr{X}$, $\pi_0^{ch}(\mathscr{X})$ denotes the sheaf $\pi_0^s(\mathrm{Sing}^{\mathbb{A}^1}(\mathscr{X}))$, where $\mathrm{Sing}^{\mathbb{A}^1}(\mathscr{X})$ is Suslin–Voevodsky's singular construction, that is, the simplicial Nisnevich sheaf given by

$$\mathrm{U} \mapsto \mathrm{Hom}_{\mathsf{Sm}_k}(\mathrm{U} \times \Delta^\bullet, \mathscr{X}).$$

Then the proof of the following theorem is a major goal of the paper. Recall that by the unstable $\mathbb{A}^1$-0-connectivity theorem, given any space $\mathscr{X}$, there is an epimorphism $\pi_0^{ch}(\mathscr{X}) \to \pi_0^{\mathbb{A}^1}(\mathscr{X})$ of Nisnevich sheaves.

Theorem 3.2 ([AM11, Theorem 2.4.3])**.** *Let* X *be a proper scheme of finite type over* k*. Then the canonical epimorphism* $\pi_0^{ch}(\mathrm{X}) \to \pi_0^{\mathbb{A}^1}(\mathrm{X})$ *induces a bijection* $\pi_0^{ch}(\mathrm{X})(\mathrm{L}) \to \pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L})$ *for any finitely generated separable extension* L/k*.*

Several corollaries are mentioned.

Corollary 3.3 ([AM11, Corollary 2.4.4])**.** *If* X *is a smooth proper scheme over* k*, then* X *is* $\mathbb{A}^1$*-chain connected if, and only if,* X *is* $\mathbb{A}^1$*-chain connected.*

Proof. As we have already seen in the last talk (see [AM11, p. 2.2.7]), if X is $\mathbb{A}^1$-chain connected, then X is $\mathbb{A}^1$-connected (in fact, no smoothness hypothesis is required for this implication). Reciprocally, the above theorem shows that if X is $\mathbb{A}^1$-connected, then $\pi_0(\mathrm{X})(\mathrm{L}) = \mathrm{X}(\mathrm{L})/\!\sim$ is trivial for all L/k finitely generated and separable. Hence X is $\mathbb{A}^1$-chain connected by definition. □

The authors also make the following claim:

> If X is smooth and proper over k, then X is separably rationally connected if, and only if, X is weakly $\mathbb{A}^1$-connected.

Again, as observed above for one direction, either sense of this equivalence likely appeals to results about birational geometry which we do not know.

Corollary 3.4 ([AM11, Corollary 2.4.6])**.** *Let* k *be a field of characteristic* 0*. Let* X *and* X′ *be* k*-birationally equivalent smooth proper varieties. Then* X *is* $\mathbb{A}^1$*-connected if, and only if,* X′ *is* $\mathbb{A}^1$*-connected.*

Proof. This follows from Example 2.9 and Theorem 2.10. □

The following is claimed to be a corollary of the above theorem but again, it seems to rely on properties of separably rationally connected varieties that are unknown to the author; *e.g.* that if k is of characteristic 0, then $\mathbb{A}^1$-chain connected varieties should be separably rationally connected. As such, we were unable to verify it.

> Let X be a smooth proper k-variety, with k algebraically closed of characteristic 0. Assume that $\dim \mathrm{X} \leqslant 2$. Then X is $\mathbb{A}^1$-connected if, and only if, X is rational.

The fact that separably rationally connected surfaces are rational if k is algebraically closed is an exercise in [Kol96] is indeed crucial to the proof given in [AM11].

The final claim [AM11, Corollary 2.4.9] has to do with compactifications. To this end, recall the R-relation of Manin defined on the set of L-points of a scheme X. Let L/k be a seperable, finite-type extension. We say that points x and x' of X(L) are R-*equivalent* if there exists an open subscheme U of $\mathbb{P}^1_{\mathrm{L}}$ containing 0 and ∞ and an L-morphism $h : \mathrm{U} \to \mathrm{X}$ such that $u(0) = x$ and $u(\infty) = 1$. We then denote by R the equivalence relation on X(L) generated by R-equivalence: X(L)/R is the set of R-*equivalence classes* of L-points of X. Clearly, $\mathbb{A}^1$-equivalent L-points are R-equivalent, hence a surjective map $\mathrm{X(L)}/\!\sim \,\to \mathrm{X(L)/R}$. This map is a bijection if X is proper over k. In any case, we have a commutative diagram:

$$\begin{array}{ccccc} & \swarrow & \mathrm{X(L)} \\ & & \downarrow & \searrow \\ \mathrm{X(L)/R} & \longleftarrow & \mathrm{X(L)}/\!\sim & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L}) \end{array}$$

where the vertical and left diagonal maps are quotient maps, hence surjective maps, and the right horizontal map is known to be surjective. In particular, the right diagonal map is surjective.

Corollary 3.5 ([AM11, Corollary 2.4.9])**.** *Let k be a field of characteristic 0. Let X be an object of Sm_k and let $j : \mathrm{X} \hookrightarrow \overline{\mathrm{X}}$ be an open immersion into a smooth proper variety. Then for any finitely generated, separable extension L/k, the image of the map $\pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L}) \to \pi_0^{\mathbb{A}^1}(\overline{\mathrm{X}})(\mathrm{L}) = \overline{\mathrm{X}}(\mathrm{L})/\!\sim$ coincides with X(L)/R. In particular, for any $\mathrm{X} \in \mathsf{Sm}_k$, the map $\mathrm{X(L)} \to \mathrm{X(L)/R}$ factors through the surjective map $\mathrm{X(L)} \to \pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L})$.*

Proof. There is a commutative diagram:

$$\begin{array}{ccccc} & \overset{p_{\mathrm{R}}}{\swarrow} & \mathrm{X(L)} & \overset{e}{\searrow} \\ & & \downarrow \\ \mathrm{X(L)/R} & \longleftarrow & \mathrm{X(L)}/\!\sim & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L}) \\ \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \overline{\mathrm{X}}(\mathrm{L})/\mathrm{R} & \underset{q_{\mathrm{R}}}{\longleftarrow} & \overline{\mathrm{X}}(\mathrm{L})/\!\sim & \overset{\rho}{\longrightarrow} & \pi_0^{\mathbb{A}^1}(\overline{\mathrm{X}})(\mathrm{L}) \end{array}$$

induced by functoriality of all the constructions involved. We are trying to identify the image of the right vertical map. Note that Theorem 3.2 states that the bottom right horizontal map is a bijection and since $\overline{X}$ is proper over k, the bottom left horizontal map is actually a bijection. Hence all bottom horizontal maps are bijections and this ensures that given $f \in \pi_0^{\mathbb{A}^1}(\mathrm{X})(\mathrm{L})$, $j_*f = j_*\xi$ where $\xi = p_{\mathrm{R}}(x)$ for any $x \in \mathrm{X(L)}$ such that $e(x) = f$, modulo these bijections, which establishes the claim. □

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