Smooth varieties up to $\mathbb{A}^1$-homotopy and algebraic *h*-cobordisms, after Asok-Morel

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23 February 2023

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  (ii) Classify n-dimensional manifold up to homotopy equivalence.
- Problems in algebraic geometry:
  (i') Classify smooth proper varieties over a fixed field k up to isomorphism.
  (ii') Classify smooth proper k-varieties up to $\mathbb{A}^1$-weak equivalence.

I. Aspects of homotopy theory for schemes

- Let $(\mathbf{Sm}_k)_{\text{ét}}$ and $(\mathbf{Sm}_k)_{Nis}$ denote the category of smooth k-schemes endowed with the structure of a site using either the étale or Nisnevich topology.

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- We have functors

$$\begin{aligned} \alpha_* &: \mathcal{S}pc_k^{\text{ét}} \longrightarrow \mathcal{S}pc_k, \text{ and} \\ \alpha^* &: \mathcal{S}pc_k \longrightarrow \mathcal{S}pc_k^{\text{ét}} \end{aligned} \tag{2}$$

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Write W_s, C_s and F_s for the resulting classes of morphisms.

Definition

- The τ-*simplicial homotopy category*, denoted $\mathcal{H}_s^\tau(k)$, is defined by

$$\mathcal{H}_s^\tau(k) := \mathcal{S}pc_k^\tau[\mathrm{W}_s^{-1}]. \quad (3)$$

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- Similarly:

$$[(\mathcal{X}, x), (\mathcal{Y}, y)]_{s,\tau}.$$

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An object $\mathcal{X} \in \mathcal{S}pc_k^{\tau}$ is called τ-$\mathbb{A}^1$-*local* if, for any object $\mathcal{Y} \in \mathcal{S}pc_k^{\tau}$, the canonical map

$$[\mathcal{Y}, \mathcal{X}]_{\tau,s} \longrightarrow [\mathcal{Y} \times \mathbb{A}^1, \mathcal{X}]_{\tau,s}, \tag{5}$$

induced by pullback along the projection $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$, is a bijection.

Definition

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{S}pc_k^\tau$ is called

- a τ-$\mathbb{A}^1$-*weak equivalence* if for any $\mathbb{A}^1$-local $\mathcal{Z} \in \mathcal{S}pc_k^\tau$ the map

$$f^* : [\mathcal{Y}, \mathcal{Z}]_{s,\tau} \longrightarrow [\mathcal{X}, \mathcal{Z}]_{s,\tau} \tag{6}$$

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- an $\mathbb{A}^1$-cofibration if it is a simplicial cofibration (i.e., a monomorphism), and
- a τ-$\mathbb{A}^1$-fibration if it has the right lifting property with respect to $\mathbb{A}^1$-acyclic cofibrations, i.e., those maps that are simultaneously $\mathbb{A}^1$-cofibrations and τ-$\mathbb{A}^1$-weak equivalences.

- A space $\mathcal{X} \in \mathcal{S}pc_k^{\tau}$ is $\mathbb{A}^1$-*fibrant* if the map $\mathcal{X} \to *$ is an $\mathbb{A}^1$-fibration.

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- More generally, the morphism $\mathcal{X} \to *$ can be factored functorially as an $\mathbb{A}^1$-acyclic cofibration followed by an $\mathbb{A}^1$-fibration.
- Thus, we obtain an $\mathbb{A}^1$-fibrant resolution functor, i.e., a pair $(Ex_{\tau,\mathbb{A}^1}, \theta_{\tau,\mathbb{A}^1})$ consisting of an endo-functor

$$Ex_{\tau,\mathbb{A}^1} : \mathcal{S}pc_k^{\tau} \to \mathcal{S}pc_k^{\tau}$$

and a natural transformation

$$\theta_{\tau,\mathbb{A}^1} : Id \to Ex_{\tau,\mathbb{A}^1}$$

such that for any $\mathcal{X} \in \mathcal{S}pc_k^{\tau}$, the map $\mathcal{X} \to Ex_{\tau,\mathbb{A}^1}(\mathcal{X})$ is an $\mathbb{A}^1$-acyclic cofibration with $Ex_{\tau,\mathbb{A}^1}(\mathcal{X})$ an $\mathbb{A}^1$-fibrant space.

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- $$\mathcal{H}_\bullet^{\text{ét}}(k) = \mathcal{S}pc_{k,\bullet}^{\text{ét}}[W_{\mathbb{A}^1}^{-1}].$$
- $$L_{\mathbb{A}^1} : \mathcal{S}pc_k^\tau \longrightarrow \mathcal{S}pc_k^\tau \tag{7}$$

  for the left derived functor of *Id* and call it the $\mathbb{A}^1$-*localization functor*.

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  for the left derived functor of *Id* and call it the $\mathbb{A}^1$*-localization functor*.
- $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1}$ (resp. $[\mathcal{X}, \mathcal{Y}]_{\mathbb{A}^1,\text{ét}}$) for the set of morphisms computed in $\mathcal{H}(k)$ (resp. $\mathcal{H}^{\text{ét}}(k)$).

Connectedness in $\mathbb{A}^1$-homotopy theory

Definition

Suppose $\mathcal{X} \in \mathcal{S}pc_k$ (resp. $\mathcal{S}pc_k^{\text{ét}}$).
The *sheaf of (étale) simplicial connected components of* $\mathcal{X}$, denoted

$$\pi_0^s(\mathcal{X})$$

(resp. $\pi_0^{s,\text{ét}}(\mathcal{X})$), is the (étale) sheaf associated with the presheaf

$$U \mapsto [U, \mathcal{X}]_s$$

(resp. $U \mapsto [U, \mathcal{X}]_{s,\text{ét}}$) for $U \in \mathbf{Sm}_k$.

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- The *sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$*, denoted $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, is the sheaf associated with the presheaf

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  for $U \in \mathcal{S}m_k$.

- Similarly, for $\mathcal{X} \in \mathcal{S}pc_k^{\text{ét}}$, the *sheaf of étale $\mathbb{A}^1$-connected components*, denoted $\pi_0^{\mathbb{A}^1,\text{ét}}(\mathcal{X})$, is the étale sheaf associated with the presheaf

$$U \longmapsto [U, \mathcal{X}]_{\mathbb{A}^1,\text{ét}} \tag{9}$$

  for $U \in \mathcal{S}m_k$.

Remark

Suppose $\mathcal{X} \in \mathcal{S}pc_k$ (resp. $\mathcal{S}pc_k^{\text{ét}}$). If $L_{\mathbb{A}^1}(\mathcal{X})$ denotes the $\mathbb{A}^1$-localization functor, then one has by definition

$$\pi_0^s(L_{\mathbb{A}^1}(\mathcal{X})) = \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

(resp. $\pi_0^{s,\text{ét}}(L_{\mathbb{A}^1}(\mathcal{X}) = \pi_0^{\mathbb{A}^1,\text{ét}}(\mathcal{X}))$.

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3. These two observations allow us to define $\mathbb{A}^1$-homotopic notions of connectedness.
4. The presheaf $U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1}$ is $\mathbb{A}^1$-invariant but not $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (see [Ayoub]).

Definition

1. $\mathcal{X} \in \mathcal{S}pc_k$ (resp. $\mathcal{S}pc_k^{\text{ét}}$) is (étale) $\mathbb{A}^1$-*connected* if the canonical morphism $\mathcal{X} \to \operatorname{Spec} k$ induces an isomorphism of sheaves

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is an isomorphism on sections over separably closed extensions L/k.
3. $\mathcal{X}$ in $\mathcal{S}pc_k$ (resp. $\mathcal{S}pc_k^{\text{ét}}$) is *(étale)* $\mathbb{A}^1$-*disconnected* if it is not (étale) $\mathbb{A}^1$-connected.

Unstable $\mathbb{A}^1$-0-connectivity theorem

Suppose $\mathcal{X} \in \mathcal{S}pc_k$ (resp. $\mathcal{S}pc_k^{\text{ét}}$). The canonical map

$$\mathcal{X} \to Ex_{\mathbb{A}^1}(\mathcal{X})$$

(resp. $\mathcal{X} \to Ex_{\text{ét},\mathbb{A}^1}(\mathcal{X})$) induces an epimorphism

$$\pi_0^s(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

(resp. $\pi_0^{s,\text{ét}}(\mathcal{X}) \to \pi_0^{\mathbb{A}^1,\text{ét}}(\mathcal{X})$).

Remark

- Suppose X is a smooth $\mathbb{A}^1$-connected k-scheme. Since $\operatorname{Spec} k$ is Henselian local, the map $X(\operatorname{Spec} k) \to \pi_0^{\mathbb{A}^1}(X)(\operatorname{Spec} k)$ is surjective, and we conclude that X necessarily has a k-rational point.

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- The corresponding statement for smooth étale $\mathbb{A}^1$-connected schemes is false, i.e., smooth étale $\mathbb{A}^1$-connected k-schemes need not have a k-rational point if k is not separably closed.

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- If M is a manifold, we can study M by analyzing each connected component separately, since each such component will again be a manifold.
- Not true in $\mathbb{A}^1$-homotopy theory.

Definition

A scheme $X \in \mathcal{S}m_k$ is called $\mathbb{A}^1$-*rigid* if for every $U \in \mathcal{S}m_k$, the map

$$X(U) \longrightarrow X(U \times \mathbb{A}^1) \tag{10}$$

induced by pullback along the projection $U \times \mathbb{A}^1 \to U$ is a bijection.

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Lemma

If $X \in \mathcal{S}m_k$ is $\mathbb{A}^1$-rigid, then for any $U \in \mathbf{Sm}_k$ the canonical maps

$$\begin{aligned} X(U) &\longrightarrow [U, X]_{\mathbb{A}^1}, \text{ and} \\ X(U) &\longrightarrow [U, X]_{\mathbb{A}^1, \text{ét}} \end{aligned} \tag{11}$$

are bijections. Consequently, the canonical map $X \to \pi_0^{\mathbb{A}^1}(X)$ (resp. $X \to \pi_0^{\mathbb{A}^1, \text{ét}}(X)$) is an isomorphism of (étale) sheaves.

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- Any 0-dimensional scheme over a field k is $\mathbb{A}^1$-rigid.
- Abelian k-varieties are $\mathbb{A}^1$-rigid.
- Smooth complex varieties that can be realized as quotients of bounded Hermitian symmetric domains by actions of discrete groups are also $\mathbb{A}^1$-rigid.
- Moreover, one can produce new examples by taking (smooth) subvarieties or taking products.

Lemma

A smooth k-scheme X is $\mathbb{A}^1$-rigid if and only if for every finitely generated separable extension L/k the map

$$X(L) \longrightarrow X(\mathbb{A}^1_L) \tag{12}$$

induced by the projection $\mathbb{A}^1_L \to \operatorname{Spec} L$ is a bijection.

$\mathbb{A}^1$-homotopy classification of curves

- Two smooth proper curves of genus $g \geq 1$ are $\mathbb{A}^1$-weakly equivalent if and only if they are isomorphic.
- A smooth proper curve is $\mathbb{A}^1$-connected if and only if it is isomorphic to $\mathbb{P}^1$.

Idea of proof

- Any curve of genus $g \geq 1$ is $\mathbb{A}^1$-rigid.
- $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid.
- If $g = 0$ and $C(\emptyset) \neq 0$, then $C \simeq \mathbb{P}^1$.

Definition

Let

- - $X \in \mathbf{Sm}_k$,
  - L a finitely generated separable extension of k,
  - points $x_0, x_1 \in X(L)$.

  An *elementary* $\mathbb{A}^1$*-equivalence between* x_0 *and* x_1 is a morphism $f : \mathbb{A}^1 \to X$ such that

  $$f(0) = x_0 \text{ and } f(1) = x_1.$$

- Two points $x, x' \in X(L)$ are $\mathbb{A}^1$*-equivalent* if they are equivalent with respect to the equivalence relation generated by elementary $\mathbb{A}^1$-equivalence.
- Write $X(L)/\sim$ for the quotient of the set of L-rational points for the above equivalence relation and refer to this quotient as the set of $\mathbb{A}^1$*-equivalence classes of* L*-points*.

Definition

We say that $X \in \mathcal{S}m_k$ is *(weakly)* $\mathbb{A}^1$*-chain connected* if for every finitely generated separable field extension L/k (resp. separably closed field extension) the set of $\mathbb{A}^1$-equivalences classes of L-points $X(L)/_\sim$ consists of exactly 1 element.

Remark

- Two k-points in $X \in \mathbf{Sm}_k$ are called *directly R-equivalent* if there exists a morphism from an open subscheme of $\mathbb{P}^1$ to X whose image contains the given points.

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- If X is a smooth proper k-variety, then $\mathbb{A}^1$-chain connectedness of X is equivalent to the notion of *separable R-triviality* of X.

Conjecture

$\mathbb{A}^1$-chain connectedness is equivalent to retract k-rationality.

- The *algebraic n-simplex* is the smooth affine k-scheme

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- Note that $\Delta^n_{\mathbb{A}^1}$ is non-canonically isomorphic to $\mathbb{A}^n_k$.
- Given $X \in \mathcal{S}m_k$, let $Sing_*^{\mathbb{A}^1}(X)$ (resp. $Sing_*^{\mathbb{A}^1,\text{ét}}(X)$) denote the Suslin-Voevodsky singular construction of X, i.e., the (étale) simplicial sheaf defined by

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- By construction, there is a canonical morphism $X \to Sing_*^{\mathbb{A}^1}(X)$ (resp. $X \to Sing_*^{\mathbb{A}^1,\text{ét}}(X)$) that is an $\mathbb{A}^1$-weak equivalence (in the étale topology).

Definition

For $X \in \mathcal{S}m_k$, set

$$\pi_0^{ch}(X) := \pi_0^s(Sing_*^{\mathbb{A}^1}(X)), \tag{15}$$

and

$$\pi_0^{ch,\text{ét}}(X) := \pi_0^{s,\text{ét}}(Sing_*^{\mathbb{A}^1,\text{ét}}(X)). \tag{16}$$

We refer to the sheaf $\pi_0^{ch}(X)$ (resp. the étale sheaf $\pi_0^{ch,\text{ét}}(X)$) as the *sheaf of (étale) $\mathbb{A}^1$-chain connected components of X.*

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We have (up to sheafification)

$$\pi_0^{ch}(X)(U) = \frac{X(U)}{{}_{\sigma_0}X(U \times \mathbb{A}^1)_{\sigma_1}},$$

the quotient by the relation generated by $\sigma_0(t) \sim \sigma_1(t)$, where σ_0 and σ_1 are induced by the $0-$ and $1-$sections $U \to U \times \mathbb{A}^1$.

Lemma

Suppose $X \in \mathbf{Sm}_k$. The maps

$$\begin{aligned} \pi_0^{ch}(X) &\longrightarrow \pi_0^{\mathbb{A}^1}(X) \\ \pi_0^{ch,\text{ét}}(X) &\longrightarrow \pi_0^{\mathbb{A}^1,\text{ét}}(X) \end{aligned} \tag{17}$$

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Proof.

Since the canonical map $X \to Sing_*^{\mathbb{A}^1}(X)$ (resp. $X \to Sing_*^{\mathbb{A}^1,\text{ét}}(X)$) is an $\mathbb{A}^1$-weak equivalence (in the étale topology), the result follows immediately from the unstable $\mathbb{A}^1$-connectivity theorem applied to $Sing_*^{\mathbb{A}^1}(X)$ or $Sing_*^{\mathbb{A}^1,\text{ét}}(X)$. □

Corollary

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- If L/k is a finitely generated separable extension (or separably closed extension) and $X(L)/_{\sim} = *$,
  then $\pi_0^{\mathbb{A}^1}(X)(L) = *$.

Corollary

- Suppose X is a smooth variety over a field k.
- If L/k is a finitely generated separable extension (or separably closed extension) and $X(L)/_{\sim} = *$,
  then $\pi_0^{\mathbb{A}^1}(X)(L) = *$.
- Thus, if X is weakly $\mathbb{A}^1$-chain connected, it is weakly $\mathbb{A}^1$-connected.

Proposition (Morel)

If $X \in \mathbf{Sm}_k$ is $\mathbb{A}^1$-chain connected, then X is $\mathbb{A}^1$-connected.

Idea of proof

- Nisnevich topology is crucial.
- To check triviality of all stalks, one shows $\pi_0^{\mathbb{A}^1}(X)(S) = *$ for S any henselian local scheme.
- Chain connectedness implies that the sections over the generic point of S are trivial and also that the sections over the closed point are trivial.
- Use a sandwiching argument to establish that sections over S are also trivial (thanks to the homotopy purity theorem).

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Conjecture

The epimorphism $\pi_0^{ch}(X) \to \pi_0^{\mathbb{A}^1}(X)$ is always an isomorphism. In particular, an object $X \in \mathbf{Sm}_k$ is $\mathbb{A}^1$-chain connected if and only if it is $\mathbb{A}^1$-connected.

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5. True for more general examples, see [BS20].
6. $\mathbf{Sing}_*^{\mathbb{A}^1}(X)$ is not always $\mathbb{A}^1$-local, see [BHS15] and [BS20].

Definition

An n-dimensional smooth k-variety X is *covered by affine spaces* if X admits an open affine cover by finitely many copies of $\mathbb{A}^n_k$ such that the intersection of any two copies of $\mathbb{A}^n_k$ has a k-point.

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Lemma

If X is a smooth k-variety that is covered by affine spaces, then X is $\mathbb{A}^1$-chain connected.

$$
\begin{array}{ccccc}
\text{covered by affine spaces} & & & & \\
\Downarrow & & & & \\
\mathbb{A}^1\text{-chain connected} & \Longrightarrow & \mathbb{A}^1\text{-connected} & \Longrightarrow & \text{étale } \mathbb{A}^1\text{-connected} \\
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\text{weakly } \mathbb{A}^1\text{-chain connected} & \Longrightarrow & \text{weakly } \mathbb{A}^1\text{-connected} & &
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$$

Thank you!