Smooth varieties up to \mathbb{A}^1 -homotopy and algebraic *h*-cobordisms, after Asok-Morel

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- Problems in algebraic geometry:
 - (i') Classify smooth proper varieties over a fixed field k up to isomorphism.
 - (ii') Classify smooth proper k-varieties up to \mathbb{A}^1 -weak equivalence.

I. Aspects of homotopy theory for schemes

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- Write Spc_k^{τ} for the category of simplicial sheaves on Sm_k equipped with the topology τ where τ denotes either the Nisnevich or étale topologies.
- We have functors

$$\begin{array}{l} \alpha_* : \mathcal{S}pc_k^{\text{\'et}} \longrightarrow \mathcal{S}pc_k, \text{ and} \\ \alpha^* : \mathcal{S}pc_k \longrightarrow \mathcal{S}pc_k^{\text{\'et}} \end{array} \tag{2}$$

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Write W_s , C_s and F_s for the resulting classes of morphisms.

• The τ -simplicial homotopy category, denoted $\mathcal{H}_{s}^{\tau}(k)$, is defined by

$$\mathcal{H}_{s}^{\tau}(k) := \mathcal{S}pc_{k}^{\tau}[\mathbb{W}_{s}^{-1}].$$
(3)

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• Given $\mathcal{X}, \mathcal{Y} \in \mathcal{S}pc_k^{ au}$ we write

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• Similarly:

 $[(\mathcal{X},x),(\mathcal{Y},y)]_{s,\tau}.$

An object $\mathcal{X} \in Spc_k^{\tau}$ is called τ - \mathbb{A}^1 -local if, for any object $\mathcal{Y} \in Spc_k^{\tau}$, the canonical map

$$[\mathcal{Y},\mathcal{X}]_{\tau,s} \longrightarrow [\mathcal{Y} \times \mathbb{A}^1,\mathcal{X}]_{\tau,s},\tag{5}$$

induced by pullback along the projection $\mathcal{X} imes \mathbb{A}^1 o \mathcal{X}$, is a bijection.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ in \mathcal{Spc}_k^{τ} is called

• a au- \mathbb{A}^1 -weak equivalence if for any \mathbb{A}^1 -local $\mathcal{Z}\in\mathcal{S}pc_k^{ au}$ the map

$$f^*: [\mathcal{Y}, \mathcal{Z}]_{s,\tau} \longrightarrow [\mathcal{X}, \mathcal{Z}]_{s,\tau}$$
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- a τ - \mathbb{A}^1 -fibration if it has the right lifting property with respect to \mathbb{A}^1 -acyclic cofibrations, i.e., those maps that are simultaneously \mathbb{A}^1 -cofibrations and τ - \mathbb{A}^1 -weak equivalences.

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- More generally, the morphism $\mathcal{X} \to *$ can be factored functorially as an \mathbb{A}^1 -acyclic cofibration followed by an \mathbb{A}^1 -fibration.
- Thus, we obtain an \mathbb{A}^1 -fibrant resolution functor, i.e., a pair $(Ex_{\tau,\mathbb{A}^1}, \theta_{\tau,\mathbb{A}^1})$ consisting of an endo-functor

$$Ex_{\tau,\mathbb{A}^1}: Spc_k^{\tau} \to Spc_k^{\tau}$$

and a natural transformation

$$heta_{ au,\mathbb{A}^1}: \mathit{Id} o \mathit{Ex}_{ au,\mathbb{A}^1}$$

such that for any $\mathcal{X} \in Spc_k^{\tau}$, the map $\mathcal{X} \to Ex_{\tau,\mathbb{A}^1}(\mathcal{X})$ is an \mathbb{A}^1 -acyclic cofibration with $Ex_{\tau,\mathbb{A}^1}(\mathcal{X})$ an \mathbb{A}^1 -fibrant space.

Write

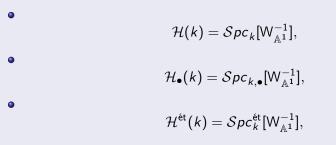
$\mathcal{H}(k) = \mathcal{S}pc_k[\mathsf{W}_{\mathbb{A}^1}^{-1}],$

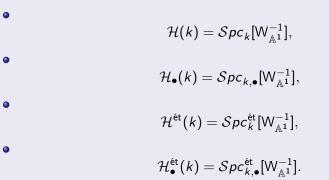
Write

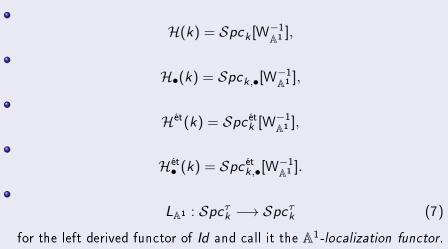
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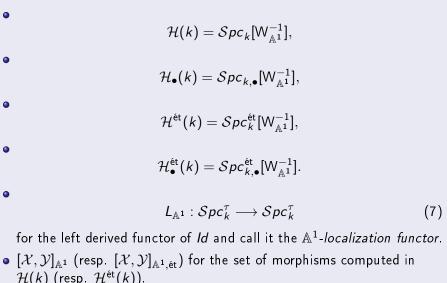
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Connectedness in \mathbb{A}^1 -homotopy theory

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Suppose $\mathcal{X} \in Spc_k$ (resp. $Spc_k^{\text{ét}}$). The sheaf of (étale) simplicial connected components of \mathcal{X} , denoted

 $\pi_0^s(\mathcal{X})$

(resp. $\pi_0^{s,\text{\'et}}(\mathcal{X})$), is the (étale) sheaf associated with the presheaf

 $U\mapsto [U,\mathcal{X}]_s$

(resp. $U \mapsto [U, \mathcal{X}]_{s, \text{\'et}}$) for $U \in \mathbf{Sm}_k$.

Suppose $\mathcal{X} \in Spc_k$.

The sheaf of A¹-connected components of X, denoted π₀^{A¹}(X), is the sheaf associated with the presheaf

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• The sheaf of \mathbb{A}^1 -connected components of \mathcal{X} , denoted $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, is the sheaf associated with the presheaf

$$U \longmapsto [U, \mathcal{X}]_{\mathbb{A}^1},$$
 (8)

for $U \in Sm_k$.

 Similarly, for X ∈ Spc^{ét}_k, the sheaf of étale A¹-connected components, denoted π₀^{A¹,ét}(X), is the étale sheaf associated with the presheaf

$$U \longmapsto [U, \mathcal{X}]_{\mathbb{A}^1, \text{\'et}} \tag{9}$$

for $U \in Sm_k$.

Suppose $\mathcal{X} \in Spc_k$ (resp. $Spc_k^{\text{ét}}$). If $L_{\mathbb{A}^1}(\mathcal{X})$ denotes the \mathbb{A}^1 -localization functor, then one has by definition

$$\pi^s_0(L_{\mathbb{A}^1}(\mathcal{X}))=\pi^{\mathbb{A}^1}_0(\mathcal{X})$$

(resp. $\pi_0^{s,\text{\'et}}(L_{\mathbb{A}^1}(\mathcal{X}) = \pi_0^{\mathbb{A}^1,\text{\'et}}(\mathcal{X})).$

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- The final object Spec k in the category Spc_k (resp. Spc^{ét}_k) is simplicially fibrant and A¹-local.
- 2 Thus $\pi_0^{\mathbb{A}^1}(\operatorname{Spec} k) = \operatorname{Spec} k$ and $\pi_0^{\mathbb{A}^1, \operatorname{\acute{e}t}}(\operatorname{Spec} k) = \operatorname{Spec} k$.

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- These two observations allow us to define A¹-homotopic notions of connectedness.
- The presheaf $U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1}$ is \mathbb{A}^1 -invariant but not $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (see [Ayoub]).

• $\mathcal{X} \in Spc_k$ (resp. $Spc_k^{\text{ét}}$) is (étale) \mathbb{A}^1 -connected if the canonical morphism $\mathcal{X} \to Spec k$ induces an isomorphism of sheaves

$$\pi_{\mathsf{0}}^{\mathbb{A}^{\mathsf{1}}}(\mathcal{X}) \stackrel{\sim}{ o} \operatorname{\mathsf{Spec}} k$$

(resp. isomorphism of étale sheaves $\pi_0^{\mathbb{A}^1,\text{\'et}}(\mathcal{X}) \xrightarrow{\sim} \operatorname{Spec} k$).

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X in Spc_k (resp. Spc^{ét}_k) is (étale) A¹-disconnected if it is not (étale) A¹-connected.

Unstable \mathbb{A}^1 -0-connectivity theorem

Suppose $\mathcal{X} \in \mathcal{Spc}_k$ (resp. $\mathcal{Spc}_k^{\text{\'et}}$). The canonical map

$$\mathcal{X} \to Ex_{\mathbb{A}^1}(\mathcal{X})$$

(resp. $\mathcal{X} \to E_{x_{\acute{et},\mathbb{A}^1}}(\mathcal{X})$) induces an epimorphism

$$\pi_0^s(\mathcal{X}) o \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

(resp. $\pi_0^{s,\text{\'et}}(\mathcal{X}) \to \pi_0^{\mathbb{A}^1,\text{\'et}}(\mathcal{X})$).

• Suppose X is a smooth \mathbb{A}^1 -connected k-scheme. Since Spec k is Henselian local, the map $X(\operatorname{Spec} k) \to \pi_0^{\mathbb{A}^1}(X)(\operatorname{Spec} k)$ is surjective, and we conclude that X necessarily has a k-rational point.

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- The corresponding statement for smooth étale A¹-connected schemes is false, i.e., smooth étale A¹-connected k-schemes need not have a k-rational point if k is not separably closed.

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- If *M* is a manifold, we can study *M* by analyzing each connected component separately, since each such component will again be a manifold.
- Not true in \mathbb{A}^1 -homotopy theory.

A scheme $X \in Sm_k$ is called \mathbb{A}^1 -rigid if for every $U \in Sm_k$, the map

$$X(U) \longrightarrow X(U imes \mathbb{A}^1)$$
 (10)

induced by pullback along the projection $U imes \mathbb{A}^1 o U$ is a bijection.

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Lemma

If $X\in \mathcal{S}m_k$ is \mathbb{A}^1 -rigid, then for any $U\in \mathbf{Sm}_k$ the canonical maps

$$\begin{array}{l} X(U) \longrightarrow [U,X]_{\mathbb{A}^1}, \text{ and} \\ X(U) \longrightarrow [U,X]_{\mathbb{A}^1, \text{\'et}} \end{array} \tag{11}$$

are bijections. Consequently, the canonical map $X \to \pi_0^{\mathbb{A}^1}(X)$ (resp. $X \to \pi_0^{\mathbb{A}^1,\text{\'et}}(X)$) is an isomorphism of (étale) sheaves.

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- Any 0-dimensional scheme over a field k is \mathbb{A}^1 -rigid.
- Abelian k-varieties are \mathbb{A}^1 -rigid.
- Smooth complex varieties that can be realized as quotients of bounded Hermitian symmetric domains by actions of discrete groups are also \mathbb{A}^1 -rigid.
- Moreover, one can produce new examples by taking (smooth) subvarieties or taking products.

Lemma

A smooth k-scheme X is \mathbb{A}^1 -rigid if and only if for every finitely generated separable extension L/k the map

$$X(L) \longrightarrow X(\mathbb{A}^1_L) \tag{12}$$

induced by the projection $\mathbb{A}^1_L \to \operatorname{Spec} L$ is a bijection.

\mathbb{A}^1 -homotopy classification of curves

- Two smooth proper curves of genus g ≥ 1 are A¹-weakly equivalent if and only if they are isomorphic.
- A smooth proper curve is A¹-connected if and only if it is isomorphic to P¹.

Idea of proof

- Any curve of genus $g \geq 1$ is \mathbb{A}^1 -rigid.
- \mathbb{G}_m is \mathbb{A}^1 -rigid.
- If g=0 and $C(\emptyset)
 eq 0$, then $C\simeq \mathbb{P}^1$.



Let

- • $X \in \mathbf{Sm}_k$,
 - L a finitely generated separable extension of k,
 - points $x_0, x_1 \in X(L)$.

An elementary \mathbb{A}^1 -equivalence between x_0 and x_1 is a morphism $f: \mathbb{A}^1 \to X$ such that

 $f(0) = x_0$ and $f(1) = x_1$.

- Two points $x, x' \in X(L)$ are \mathbb{A}^1 -equivalent if they are equivalent with respect to the equivalence relation generated by elementary \mathbb{A}^1 -equivalence.
- Write X(L)/~ for the quotient of the set of L-rational points for the above equivalence relation and refer to this quotient as the set of ¹-equivalence classes of L-points.

We say that $X \in Sm_k$ is (weakly) \mathbb{A}^1 -chain connected if for every finitely generated separable field extension L/k (resp. separably closed field extension) the set of \mathbb{A}^1 -equivalences classes of L-points $X(L)/\sim$ consists of exactly 1 element.

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- We say that X is separably R-trivial if for every finitely generated separable extension field L of k, X(L)/R = *.

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- We say that X is separably R-trivial if for every finitely generated separable extension field L of k, X(L)/R = *.
- If X is a smooth proper k-variety, then A¹-chain connectedness of X is equivalent to the notion of separable R-triviality of X.

Conjecture

 \mathbb{A}^1 -chain connectedness is equivalent to retract k-rationality.



$$\Delta_{\mathbb{A}^1}^n := \operatorname{Spec} k[x_0, \dots, x_n] / (\sum_{i=0}^n x_i - 1).$$
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• Note that $\Delta_{\mathbb{A}^1}^n$ is non-canonically isomorphic to \mathbb{A}_k^n .

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- Note that $\Delta_{\mathbb{A}^1}^n$ is non-canonically isomorphic to \mathbb{A}_k^n .
- Given $X \in Sm_k$, let $Sing_*^{\mathbb{A}^1}(X)$ (resp. $Sing_*^{\mathbb{A}^1,\text{\'et}}(X)$) denote the Suslin-Voevodsky singular construction of X, i.e., the ('etale) simplicial sheaf defined by

$$U \mapsto Hom_{\mathbf{Sm}_k}(\Delta^{\bullet}_{\mathbb{A}^1} \times U, X).$$
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By construction, there is a canonical morphism X → Sing^{A1}_{*}(X) (resp. X → Sing^{A1,ét}_{*}(X)) that is an A¹-weak equivalence (in the étale topology).

For $X \in Sm_k$, set

$$\pi_0^{ch}(X) := \pi_0^s(Sing_*^{\mathbb{A}^1}(X)), \tag{15}$$

and

$$\pi_0^{ch,\acute{\text{e}t}}(X) := \pi_0^{s,\acute{\text{e}t}}(Sing_*^{\mathbb{A}^1,\acute{\text{e}t}}(X)).$$
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We refer to the sheaf $\pi_0^{ch}(X)$ (resp. the étale sheaf $\pi_0^{ch,\text{ét}}(X)$) as the sheaf of (étale) \mathbb{A}^1 -chain connected components of X.

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We have (up to sheafification)

$$\pi_0^{ch}(X)(U) = rac{X(U)}{\sigma_0 X(U imes \mathbb{A}^1)_{\sigma_1}},$$

the quotient by the relation generated by $\sigma_0(t) \sim \sigma_1(t)$, where σ_0 and σ_1 are induced by the 0- and 1-sections $U \to U \times \mathbb{A}^1$.

Lemma

Suppose $X \in \mathbf{Sm}_k$. The maps

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Proof.

Since the canonical map $X \to Sing_*^{\mathbb{A}^1}(X)$ (resp. $X \to Sing_*^{\mathbb{A}^1, \text{\acute{e}t}}(X)$) is an \mathbb{A}^1 -weak equivalence (in the étale topology), the result follows immediately from the unstable \mathbb{A}^1 -connectivity theorem applied to $Sing_*^{\mathbb{A}^1, \text{\acute{e}t}}(X)$ or $Sing_*^{\mathbb{A}^1, \text{\acute{e}t}}(X)$.

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Corollary

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- If L/k is a finitely generated separable extension (or separably closed extension) and $X(L)/_{\sim} = *$, then $\pi_0^{\mathbb{A}^1}(X)(L) = *$.
- Thus, if X is weakly \mathbb{A}^1 -chain connected, it is weakly \mathbb{A}^1 -connected.

Proposition (Morel)

If $X \in \mathbf{Sm}_k$ is \mathbb{A}^1 -chain connected, then X is \mathbb{A}^1 -connected.

Idea of proof

- Nisnevich topology is crucial.
- To check triviality of all stalks, one shows π₀^{A¹}(X)(S) = ∗ for S any henselian local scheme.
- Chain connectedness implies that the sections over the generic point of S are trivial and also that the sections over the closed point are trivial.
- Use a sandwiching argument to establish that sections over S are also trivial (thanks to the homotopy purity theorem).



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Conjecture

The epimorphism $\pi_0^{ch}(X) \to \pi_0^{\mathbb{A}^1}(X)$ is always an isomorphism. In particular, an object $X \in \mathbf{Sm}_k$ is \mathbb{A}^1 -chain connected if and only if it is \mathbb{A}^1 -connected.

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- S True for more general examples, see [BS20].
- Sing^{\mathbb{A}^1}(X) is not always \mathbb{A}^1 -local, see [BHS15] and [BS20].

Definition

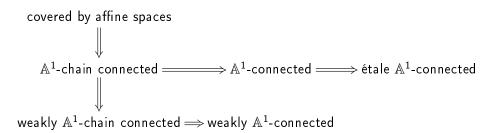
An *n*-dimensional smooth *k*-variety X is covered by affine spaces if X admits an open affine cover by finitely many copies of \mathbb{A}_k^n such that the intersection of any two copies of \mathbb{A}_k^n has a *k*-point.

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Lemma

If X is a smooth k-variety that is covered by affine spaces, then X is \mathbb{A}^1 -chain connected.



Thank you!