$\mathbb{A}^1$-connected components — 20.04.2023

§1. Strong $\mathbb{A}^1$-invariance of algebraic groups

§2. First properties of the $\mathbb{A}^1$-fundamental group

§1. k field

(Guillaume's talk)

Motivation: We saw that if $X \in Sm/k$ and $g : Z \to X \times \mathbb{A}^1$ smooth surjective st $\begin{cases} \exists\, Y \in Sm/k \text{ proper} \\ v : U \to X \text{ Nisnevich cover} \end{cases}$ st

$$\begin{tikzcd}
Z \arrow[d, "g"] \arrow[dd, bend right, "f"'] & U \times \mathbb{A}^1 \times Y \arrow[l] \arrow[d] \\
X \times \mathbb{A}^1 \arrow[d, "p"] \arrow[dl] & U \times \mathbb{A}^1 \arrow[l] \arrow[d] \\
\mathbb{A}^1 \quad X & U \arrow[l, "v"']
\end{tikzcd}$$

then f is $\mathbb{A}^1$-cobordism.

- We also saw (Rakesh's talk) that if $\mathcal{Y}$ gives rise to a non-trivial $\mathbb{A}^1$-cob., then the associated class in $H^1_{Nis}(X \times \mathbb{A}^1, \mathrm{Aut}(\mathcal{Y}))$ is not in the image of $p^*: H^1_{Nis}(X, \mathrm{Aut}(\mathcal{Y})) \to H^1_{Nis}(X \times \mathbb{A}^1, \mathrm{Aut}(\mathcal{Y}))$.

- So we want to understand when $\mathrm{Aut}(\mathcal{Y})$ is (not) strongly $\mathbb{A}^1$-invariant.

$\Rightarrow$ study strong $\mathbb{A}^1$-invariance of algebraic groups

Prop. 4.4.3 Let G be a smooth algebraic group over k of characteristic zero. Then G is étale strongly $\mathbb{A}^1$-invariant $\Leftrightarrow$ $\underbrace{G^0}_{\text{conn. comp. of id}}$ is a semi-abelian variety. In this case, G is also strongly $\mathbb{A}^1$-invariant.

To give the proof:

<u>Lemma 4.4.1</u>. Let k be perfect and $G/_k$ smooth <u>affine</u> alg. group. Then G is $\mathbb{A}^1$-invariant $\Leftrightarrow$ G^0 is a torus.

<u>Proof</u>. (<u>Recall</u> (Niels' talk) $X \in Sm/_k$ is $\mathbb{A}^1$-rigid $\Leftrightarrow$ $X(L) \simeq X(\mathbb{A}^1_L)$ $\forall\, L|k$ finitely generated separable.)

1) $\exists$ s.e.s. $1 \longrightarrow G^0 \longrightarrow G \longrightarrow \underbrace{\pi_0(G)}_{\text{group of connected components, finite}/k\text{, hence } \mathbb{A}^1\text{-invariant}} \longrightarrow 1$

$\Rightarrow$ G $\mathbb{A}^1$-invariant $\Leftrightarrow$ G^0 is

So suppose G connected.

2) $\exists$ s.e.s. $1 \longrightarrow \underbrace{R_u(G)}_{\text{unipotent radical of } G} \longrightarrow G \longrightarrow G^{red} \longrightarrow 1$

reductive

(largest connected normal unipotent subgroup of G)

(since k perfect, $R_u(G)/_k$ smooth)

Now if $R_u(G) \neq 1$, then $\exists\ 1 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \dots \trianglelefteq N_r = R_u(G)$
(connected, smooth)
with $N_{i+1}/N_i \cong \mathbb{G}_a$. Hence G is $\mathbb{A}^1$-invariant $\Rightarrow R_u(G) = 1$.

So suppose G reductive.

3) $\exists$ s.e.s. $1 \to \underbrace{R(G)}_{\substack{\text{torus, largest} \\ \text{subtorus of the center } Z(G)}} \to G \to G^{ss} \to 1$

semi-simple

We want to show $G^{ss} \neq 1 \Rightarrow G$ not $\mathbb{A}^1$-invariant.

In fact, over some finite $L|k$, $\exists\ \mathbb{A}^1_L \to G^{ss}_L$ non-constant:
(if $G^{ss} \neq 1$)

take any root subgroup $G_\alpha \hookrightarrow G^{ss}_L$, $G_\alpha \simeq \mathbb{G}_a$. centralizer
(T_L maximal torus of G^{ss}_L, $\alpha : T_L \to \mathbb{G}_{m,L}$ character
$G_\alpha :=$ {unipotent elements in the Borel of the commutator sgp of $C_G(\ker(\alpha)^\circ)$}.

So (since tori are $\mathbb{A}^1$-invariant) this concludes the proof. □

Lemma 6.6.2 Let k be perfect, G/k smooth algebraic group. Then G $\mathbb{A}^1$-invariant $\Leftrightarrow$ G^0 is a semi-abelian variety.

Proof. - We saw that we can suppose G connected.

$\exists$ s.e.s. $1 \longrightarrow G^{aff} \longrightarrow G \longrightarrow G^{ab} \longrightarrow 1$

(Chevalley dévissage)

largest connected affine sbgp of G abelian variety

By the previous lemma, G $\mathbb{A}^1$-invariant $\Rightarrow$ G^{aff} torus.

On the other hand, abelian varieties are $\mathbb{A}^1$-rigid, hence $\mathbb{A}^1$-invariant as algebraic groups. □

Proof of Prop. 6.6.3. In char 0, a smooth algebraic group G/k admitting transfers is $\mathbb{A}^1$-invariant $\Leftrightarrow$ étale strongly $\mathbb{A}^1$-invariant

(Rakesh's talk(s)). Now we saw G/k smooth algebraic group is $\mathbb{A}^1$-invariant $\Leftrightarrow$ G° semi-abelian variety, and a semi-abelian admits transfers because it is commutative.

$\hookrightarrow$ Recall how this works: $X, Y \in Sm_k$, $C(X,Y) :=$

$= \langle Z \subseteq X \times Y \mid$ finite over X, surjective over a connected comp. of $X \rangle_{\mathbb{Z}}$

· G smooth commutative alg. group $/k$, $Z \in C(X,Y)$

$\Rightarrow$ get $Z: G(Y) \to G(X)$, $y \mapsto x_Z$ $\Rightarrow$ G admits transfers.

$$\begin{tikzcd}
G \arrow[r] & G^{(d)} \arrow[r, "\mathrm{sum}"] & G \\
Y \arrow[u, "y"] \arrow[r] & Y^{(d)} \arrow[u, "y^{(d)}"] & {:= d\text{-th symmetric power}} \\
X \arrow[ur, "\exists f_Z"] \arrow[uurr, bend right, "x_Z"'] & &
\end{tikzcd}$$

$\Rightarrow$ In char 0, G étale strongly $\mathbb{A}^1$-invariant $\Leftrightarrow$ G° is a semi-abelian variety.

- To conclude, we apply the fact (Rakesh's talk) that étale strongly $\mathbb{A}^1$-invariant $\Rightarrow$ Nisnevich strongly $\mathbb{A}^1$-invariant. $\square$

- To apply this to $\mathbb{A}^1$-strong invariance of $\mathrm{Aut}(Y)$, $Y \in Sm_{/k}$ proper, observe that $\mathrm{Aut}(Y)$ is a smooth group scheme$_{/k}$ (in general, not of finite type) in this case.

§2.

$X \in Sm_k$, $x \in X(k)$, $\pi_1^{\mathbb{A}^1}(X,x) :=$ sheaf associated to

$$U \longmapsto [S^1 \wedge U_+, (X,x)]_{\mathbb{A}^1}$$

<u>Properties</u>. 1) $\pi_1^{\mathbb{A}^1}(X,x)$ strongly $\mathbb{A}^1$-invariant sheaf of groups

2) (van Kampen theorem) X $\mathbb{A}^1$-connected, covered by $\mathbb{A}^1$-connected opens U, V with $\mathbb{A}^1$-connected $U \cap V$, then

$$\pi_1^{\mathbb{A}^1}(X,x) \simeq \pi_1^{\mathbb{A}^1}(U) \underset{\pi_1^{\mathbb{A}^1}(U\cap V)}{\ast^{\mathbb{A}^1}} \pi_1(V)$$

3) If, $X, \tilde{X} \in Sm_k$ $\mathbb{A}^1$-connected, $f: \tilde{X} \to X$ $\mathbb{G}_m^r$-fibration

$\Rightarrow$ f <u>is an $\mathbb{A}^1$-fibration</u> and $\exists$ $\begin{cases} 1 \to \pi_1^{\mathbb{A}^1}(\tilde{X}) \to \pi_1^{\mathbb{A}^1}(X) \to \mathbb{G}_m^r \to 1 \\ \pi_i^{\mathbb{A}^1}(\tilde{X}) \xrightarrow{\sim} \pi_i^{\mathbb{A}^1}(X) \quad \forall i > 1 \end{cases}$

<u>Prop. 5.1.4</u> If $X \in Sm_k$ $\begin{cases} \mathbb{A}^1\text{-connected} \\ \text{proper, } \dim X > 0 \end{cases}$, then $\forall x \in X(k)$

$\pi_1^{\mathbb{A}^1}(X, x)$ is <u>non-trivial</u>.

<u>Recall</u> (Postnikov tower)

$\exists$ diagram of spaces

$$\begin{tikzcd} \cdots \arrow[r] & P^{(m)}X \arrow[r, "p^m"] & P^{(m-1)}X \arrow[r] & \cdots \\ & X \arrow[u] \arrow[ur] & & \end{tikzcd}$$

st $X \simeq \operatorname{holim}_m P^{(m)}X$
in $\mathcal{H}_s(k)$

and $\begin{cases} \pi_i^{\mathbb{A}^1}(X, x) \simeq \pi_i^{\mathbb{A}^1}(P^{(m)}X, x) \text{ for } i \leq m \\ \pi_i^{\mathbb{A}^1}(P^{(m)}X, x) \simeq 0 \ \forall i > m \\ (\text{homotopy fiber of } p^m) \simeq K(\pi_m^{\mathbb{A}^1}(X, x), m) \end{cases}$

In particular, if X $\mathbb{A}^1$-connected $\Rightarrow P^{(1)}X \simeq K(\pi_1^{\mathbb{A}^1}(X,x), 1) \simeq$

⇝ $B\pi_1^{\mathbb{A}^1}(X,x)$ ⇒ get canonical $X \xrightarrow{f} B\pi_1^{\mathbb{A}^1}(X,x)$, iso of homotopy groups for $i=0,1$.

· $\mathcal{G}r_k^{\mathbb{A}^1} := \{\mathbb{A}^1\text{-invariant sheafs of groups}\}$

Property: $[B\pi_1^{\mathbb{A}^1}(X,x), H]_{\mathbb{A}^1} \underset{(\star)}{\simeq} \mathrm{Hom}_{\mathcal{G}r_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(X,x), H)$

$H \in \mathcal{G}r_k^{\mathbb{A}^1}$

· One gets

$$\begin{tikzcd}
{[(X,x),(B\mathbb{G}_m,*)]_{\mathbb{A}^1}} \arrow[d, "\star"', "\text{(factoring through } f)"'] \arrow[d, "\text{by functoriality of the Postnikov tower}"] \arrow[dr, "\pi_1^{\mathbb{A}^1}(\dots)", "I"'] & \\
{[(B\pi_1^{\mathbb{A}^1}(X,x),x),(B\mathbb{G}_m,*)]_{\mathbb{A}^1}} \arrow[r, "\sim", "(\star)"'] & \mathrm{Hom}_{\mathcal{G}r_k^{\mathbb{A}^1}}(\pi_1^{\mathbb{A}^1}(X,x), \mathbb{G}_m,*)
\end{tikzcd}$$

· One can prove (Rakesh's talk) that I is an isomorphism.

<u>Proof of the proposition.</u>

X $\mathbb{A}^1$-connected, proper, $x \in X(k)$, $\dim X > 0$
$\in Sm_k$

$$\underset{\mathcal{G}r_k^{\mathbb{A}^1}}{\mathrm{Hom}}\left(\pi_1^{\mathbb{A}^1}(X,x), (\mathbb{G}_m, *)\right) \underset{\mathrm{I}^{-1}}{\simeq} [(X,x), (B\mathbb{G}_m, *)]_{\mathbb{A}^1} \simeq$$

$$\simeq [X, B\mathbb{G}_m]_{\mathbb{A}^1} \simeq \mathrm{Pic}(X),$$ non-trivial on a smooth, proper scheme$/k$ of $\dim > 0$.

$\mathbb{G}_m$ commutative

(Morel-Voevodsky)

In particular, $\pi_1^{\mathbb{A}^1}(X,x)$ non-trivial. $\square$

<u>Corollary</u>. Any $\mathbb{A}^1$-cobordism between $\mathbb{A}^1$-connected and simply $\mathbb{A}^1$-connected smooth proper varieties is trivial.

<u>Proposition 5.1.5</u>. If $X \in Sm_k$ $\begin{cases} \text{étale } \mathbb{A}^1\text{-connected} \\ \text{proper of } \dim > 0 \end{cases}$, then $\pi_1^{\mathbb{A}^1,\acute{e}t}(X,x) \neq$ trivial.

[<u>Corollary</u>. Same as above for étale $\mathbb{A}^1$-connected vs. simply étale $\mathbb{A}^1$-connected.

Idea: one gets as well

$$\mathrm{Hom}_{\mathcal{G}r^{\mathbb{A}^1}_k}\left(\pi_1^{\mathbb{A}^1,\text{ét}}(X,x),(\mathbb{G}_m,*)\right) \simeq [X, B\mathbb{G}_m]_{s,\text{ét}}$$

Now $[X, B\mathbb{G}_m]_{s,\text{ét}} \simeq H^1_{\text{ét}}(X,\mathbb{G}_m) \underset{\text{Hilbert's 90}}{\simeq} \mathrm{Pic}(X)$. $\square$