k perfect field
$(Sm/k$, Nisnevich topology$)$

<u>Def</u> $F \in Sh(Sm/k)$ (constant object in Δ^{op})

- $S^{pre} \in PSh(Sm/k) : U \longmapsto \dfrac{F(U)}{{}_{\sigma_0}F(U \times \mathbb{A}^1)_{\sigma_1}}$

$$u \longmapsto (u,0)$$

when $U \underset{\sigma_1}{\overset{\sigma_0}{\rightrightarrows}} U \times \mathbb{A}^1$ quotient by $h|_{\sigma_0} = h|_{\sigma_1}$

$$u \longmapsto (u,1)$$

$\forall h \in F(U \times \mathbb{A}^1)$

- $S(F) =$ sheafification of $S^{pre}(F)$
- $n \geq 1$ $S^n(F) = \underbrace{S \circ \dots \circ S(F)}_{n \text{ times}} \in Sh(Sm/k)$

Particular case : $X \in Sm/k$ $S(X) \simeq \pi_0^{ch}(X)$
(Asok-Morel)

$$\left(U \longmapsto \frac{Hom(U, X)}{{}_{\sigma_0}Hom(U \times \mathbb{A}^1, X)_{\sigma_1}} \right)^{\#}$$

Interpretation of $S(F)$ using the singular construction

$$\Delta_n = \mathrm{Spec}\, k[x_0, \dots, x_n] / (\textstyle\sum x_i = 1)$$

$$\mathrm{Sing}(F)_n = \underline{\mathrm{Hom}}(\Delta_n, F) \in Sh(Sm/k)$$

$$\left(U \longmapsto F(U \times \Delta_n)\right)^{\#}$$

$G \in \Delta^{op} PSh(Sm/k)$

$$\pi_0(G) : U \longmapsto \frac{G_0(U)}{d_0\, G_1(U)\, d_1} \qquad d_0, d_1 = \text{faces in } G_\bullet$$

$$\pi_0^{\wedge}(G) = \pi_0(G)^{\#}$$

<u>Prop</u>. $\pi_0(G)(U) = \mathrm{Hom}_{\mathcal{H}_s(k)}(U, G)$

simplicial homotopy category

In particular $S(F) = \pi_0^{\wedge}(\mathrm{Sing}\, F)$

Some properties of $S(F)$:

<u>Lemma</u>: $F \in Sh(Sm/k)$ $\quad F \xrightarrow{\text{epi}} S(F)$

<u>Proof</u>. $F \longrightarrow S^{pre} \xrightarrow{\text{epi}} S$

$$F(U) \longrightarrow \frac{F(U)}{\sigma_0\, F(U \times \mathbb{A}^1)\, \sigma_1}$$

$\square$.

The authors construct a sheaf $L(F)$

<u>Def</u>. $F \in Sh(Sm/k)$ $\quad L(F) := \varinjlim_{n \geq 1} S^n(F)$

for the transition map $S^n(F) \longrightarrow S(S^n(F))$
$$S(S^n(F)) = S^{n+1}(F)$$

<u>Thm</u> (first part of Thm 1 in [BHS])

$L(F)$ is $\mathbb{A}^1$-invariant, i.e. $\forall U \in Sm/k$,
$$L(F)(U) \xrightarrow{\simeq} L(F)(U \times \mathbb{A}^1)$$

<u>Proof</u>. • Injectivity clear since the map $U \times \mathbb{A}^1 \longrightarrow U$ has a section

• Surjectivity. $L(F)(U) = \varinjlim_n S^n(F)(U)$

Let $t \in L(F)(U \times \mathbb{A}^1)$, rep. by $t \in S^n(F)(U \times \mathbb{A}^1)$

Want: t comes from $S^{n+1}(F)(U)$

Let $m : U \times \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow U \times \mathbb{A}^1$
$$(u, x, y) \longmapsto (u, xy)$$

$m^*(t) \in S^n(F)(U \times \mathbb{A}^1 \times \mathbb{A}^1)$

$$(u,x) \longmapsto (u,0)$$
$$\begin{tikzcd} & U \times \mathbb{A}^1 \times \mathbb{A}^1 \arrow[dr, "m"] & \\ U \times \mathbb{A}^1 \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] & & U \times \mathbb{A}^1 \\ & U \times \mathbb{A}^1 \times \mathbb{A}^1 \arrow[ur, "m"'] & \end{tikzcd}$$
$$(u,x) \longmapsto (u,x)$$

$p : U \times \mathbb{A}^1 \longrightarrow U$

Therefore $\sigma_1^* m^* t = t$ and $\sigma_0^* m^* t = p^*(\overbrace{\sigma_0^* t}^{\in S^{n+1}(F)(U)})$

So in $S^{n+1}(F)(U\times \mathbb{A}^1)$, t gets identified with $p^*(\sigma_0^* t) \in p^* S^{n+1}(F)(U)$ $\square$

Prop. $\forall\, F \xrightarrow{\phi} G$ with $G \in Sh(Sm/k)$ $\mathbb{A}^1$-invariant, ϕ factors uniquely through $L(F)$

Rk. $\pi_0^{\mathbb{A}^1}(F)$ also satisfies this universal property, but $\pi_0^{\mathbb{A}^1}(F)$ is not $\mathbb{A}^1$-invariant in general.

In general:

$$\begin{tikzcd}
 & & S^2(F) \arrow[drr] & & \mathbb{A}^1\text{-inv} \\
F \arrow[r, "epi"] & S(F) \arrow[ur] \arrow[r, "epi"] & \pi_0^{\mathbb{A}^1}(F) \arrow[rr, "epi?"] & & L(F)
\end{tikzcd}$$

Prop Let $X \in \Delta^{op} Sh(Sm/k)$ and $F = \pi_0^{s}(X)$. If $\pi_0^{\mathbb{A}^1}(F)$ is $\mathbb{A}^1$-invariant, then the canonical map $\pi_0^{\mathbb{A}^1}(X) \longrightarrow \pi_0^{\mathbb{A}^1}(F)$ is an isomorphism.

Proof. Uses the unstable $\mathbb{A}^1$-connectivity theorem (Morel–Voevodsky, 3.22): $\pi_0^{s}(X) \xrightarrow{epi} \pi_0^{\mathbb{A}^1}(X)$

+ universal property of $\pi_0^{\mathbb{A}^1}(F)$ $\square$.

Question. Is $S(F)$ $\mathbb{A}^1$-invariant?

+ Explicit description of $S(F)(U)$, $U \in Sm/k$

$F \in Sh(Sm/k)$

<u>Def</u>. $F \in Sh(Sm/k)$, $U \in Sm/k$

- An $\mathbb{A}^1$-homotopy of U in F is an element of $F(U \times \mathbb{A}^1)$
- $h \in F(U \times \mathbb{A}^1)$, $t_0 = \sigma_0^*(h)$, $t_1 = \sigma_1^*(h) \in F(U)$ are said to be $\mathbb{A}^1$-homotopic.
- An $\mathbb{A}^1$-chain homotopy of U in F is a tuple $(h_1, \dots, h_n)$ with $\sigma_1^* h_i = \sigma_0^* h_{i+1}$ $\forall\, 1 \le i \le n-1$.
  Then $t_0 = \sigma_0^* h_1$ and $t_1 = \sigma_1^* h_n$ are said to be $\mathbb{A}^1$-chain homotopic.

* $t_1 = \sigma_0^* h$, $t_2 = \sigma_1^* h \in F(U)$ map to same element in $S(F)(U)$

* Conversely, if $t_1, t_2 \in F(U)$ map to same elt in $S(F)(U)$ then $\exists$ Nis-cover $V \to U$ such that $t_1|_V$ and $t_2|_V$ are $\mathbb{A}^1$-chain homotopic.

* Let $t \in S(F)(U)$ $\qquad F \xrightarrow{\varphi} S(F)$

$\Rightarrow \exists\, V \to U$ Nis-cover s.t. $t|_V$ lifts to $F(V)$.

$$
\begin{tikzcd}
 & V \times_U V \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
V \arrow[dr] & & V \arrow[dl] \\
 & U &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(V \times_U V) \arrow[r] & S(F)(V \times_U V) \\
t|_V \arrow[d, equal] \arrow[r, phantom, "\in"] & F(V) \arrow[u, "pr_i^*"] \arrow[r] & S(F)(V) \arrow[u] \\
s & F(U) \arrow[u] \arrow[r] & S(F)(U) \arrow[u] \\
 & & t \arrow[u, phantom, "\in" description]
\end{tikzcd}
$$

$\Rightarrow pr_1^*(t|_V)$ and $pr_2^*(t|_V)$ have the same image in $S(F)(V \times_U V)$

$\Rightarrow$ $(\star)$ $\begin{cases} \exists\, W \longrightarrow V \times_U V \text{ finite Nis-cover s.t.} \\ pr_1^*(s)|_W \text{ and } pr_2^*(s)|_W \text{ are } \mathbb{A}^1\text{-chain homotopic} \end{cases}$

$\rightsquigarrow$ Description of $S(F)(U)$:

$$\begin{cases} V \longrightarrow U \text{ Nis cover} \\ + \; s \in F(V) \text{ satisfying } (\star) \end{cases}$$

This leads to the notion of ghost homotopy.

<u>Def</u> $F \in Sh(Sm/k)$, $U \in Sm/k$

An $\mathbb{A}^1$-ghost homotopy of U in F is the data:

$$\mathcal{H} = (\underbrace{V \longrightarrow \mathbb{A}^1_U}_{\text{Nis cover}},\ \underbrace{W \longrightarrow V \times_{\mathbb{A}^1_U} V}_{\text{Nis cover}},\ h,\ h^W)$$

s.t. • $h \in F(V)$

• h^W is an $\mathbb{A}^1$-chain homotopy connecting the two morphisms $pr_1^*(h)|_W$ and $pr_2^*(h)|_W$

with
$$\begin{tikzcd} & & V \arrow[dr] & \\ W \arrow[r] & V \times_{\mathbb{A}^1_U} V \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & & F \\ & & V \arrow[ur] & \end{tikzcd}$$

$t_1, t_2 \in F(U)$ are said to be ghost homotopic if $\exists\, \mathcal{H} = (V, W, h, h^W)$ s.t. $\exists\, \tilde{\sigma}_0, \tilde{\sigma}_1 : U \to V$ lifting σ_0, σ_1 s.t. $\tilde{\sigma}_0^* h = t_1$ and $\tilde{\sigma}_1^* h = t_2$ with $\mathcal{H}$ as above.

NB. Ghost homotopy in $F \overset{\text{unique}}{\rightsquigarrow}$ Homotopy in $S(F)$

Lemma. $S(F) = S^2(F) \iff \forall U$ Henselian local scheme, $\forall t_1, t_2 \in F(U)$
t_1, t_2 ghost homotopic $\implies$ chain homotopic.

see Remark 3.5 in [BHS]

Def. $Z \in Sm/k$, $s_1, s_2 \in F(Z)$. $F \in Sh(Sm/k)$
$s_1 \sim_0 s_2 \iff \forall x \in Z,\ s_1 \circ x = s_2 \circ x$ in $F(k(x))$

Def. We say $F \in Sh(Sm/k)$ is almost proper if

(AP1) Let U/k smooth irreducible dim ≤ 2
Let $s \in F(U)$. Then $\exists\, \bar{U}/k$ smooth projective and $\exists\, i : U \dashrightarrow \bar{U}$ birational, $\exists\, \bar{s} \in F(\bar{U})$
$\bar{s} \circ i \sim_0 s$ on the domain of definition of i.

(AP2) Let U/k smooth irreducible curve.
Let $s_1, s_2 \in F(U)$ s.t. $s_1|_{U'} = s_2|_{U'}$ for some $U' \underset{\neq \emptyset}{\subseteq} U$. Then $s_1 \sim_0 s_2$.

Example. Proper schemes represent AP sheaves.

Lemma [Lemma 3.7 in [BHS]]. Let $F \in Sh(Sm/k)$ A.P.
Let U/k smooth curve, $x \in U(k)$, $U' = U \setminus \{x\}$
Let $f, g \in F(U)$ st. $f|_{U'}$ and $g|_{U'}$ are $\mathbb{A}^1$-chain homotopic.
Then $f \circ x$, $g \circ x \in F(k)$ are $\mathbb{A}^1$-chain homotopic.

Proof Let $h : U' \times \mathbb{A}^1 \longrightarrow F$, $\sigma_0^* h = f|_{U'}$ and $\sigma_1^* h = g|_{U'}$

(AP1) $\Rightarrow$ $\exists\, i : U' \times \mathbb{A}^1 \dashrightarrow X$ smooth proj. surface

and $\bar{h} : X \longrightarrow F$ s.t. $\bar{h} \circ i \sim_0 h$ on $W = \mathrm{Dom}(i)$.

$(U' \times \mathbb{A}^1) \setminus W$ has codim. 2 (since i defined in codim. 1)

$\Rightarrow \exists\, U'' \subseteq U'$ s.t. $U'' \times \mathbb{A}^1 \subset W$

$$U'' \longrightarrow X \qquad\qquad U'' \longrightarrow X$$
$$u \longmapsto i(u,0) \qquad\qquad u \longmapsto i(u,1)$$

They extend to $\bar{t}_0, \bar{t}_1 : \bar{U} \longrightarrow X$

$\bar{U}$: smooth compactification of U.

By assumption $\bar{h} \circ \bar{t}_0 \sim_0 f$ and $\bar{h} \circ \bar{t}_1 \sim_0 g$ on U''

$\Rightarrow \exists\, U''' \subseteq U''$ s.t. $\bar{h} \circ \bar{t}_0 = f$ and $\bar{h} \circ \bar{t}_1 = g$ on U'''

(AP 2) $\Rightarrow$ $\left| \begin{array}{l} \bar{h} \circ \bar{t}_0 \circ x = f \circ x \\ \bar{h} \circ \bar{t}_1 \circ x = g \circ x \end{array} \right.$

$\longrightarrow$ Suffices to prove that $\bar{t}_0 \circ x$ and $\bar{t}_1 \circ x$ are $\mathbb{A}^1$-chain homotopic.

By resolution of indeterminacy:

$$\exists \quad \begin{array}{c} \tilde{X} \\ \text{blowup} \downarrow \\ \bar{U} \times \mathbb{P}^1 \supseteq W \xrightarrow{\ i\ } X \end{array} \quad \exists \text{ proper birational } \tilde{X} \to X$$

and lifts $\tilde{t}_0, \tilde{t}_1 : \bar{U} \longrightarrow \tilde{X}$

Asok-Morel 6.2.9 $\Longrightarrow$ $\tilde{t}_0 \circ x$ and $\tilde{t}_1 \circ x$ are $\mathbb{A}^1$-chain homotopic. $\square$