M, N n-manifold are cobordant

W $n+1$-m st $\partial W = M \sqcup N$

W is h-cobordism if $M \hookrightarrow W$, $N \hookrightarrow W$ are homotopy eq

$n \geq 5$

h-cobordism theorem: if M, N are simply connected, then

$W \simeq M \times [0,1]$

$f : X \to \mathbb{A}^1$ is an $\mathbb{A}^1$-h-cobordism if

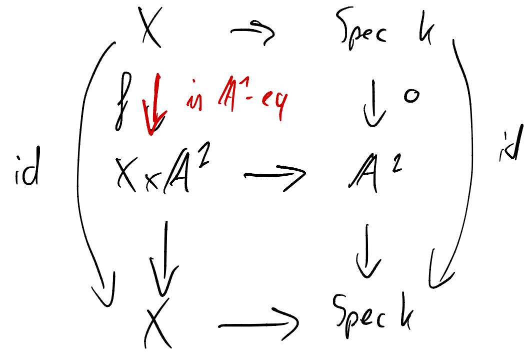
f is proper, surjective $f^{-1}(0)$, $f^{-1}(1)$ are smooth

$\phantom{f^{-1}(0)} \; = X_0 \qquad = X_1$

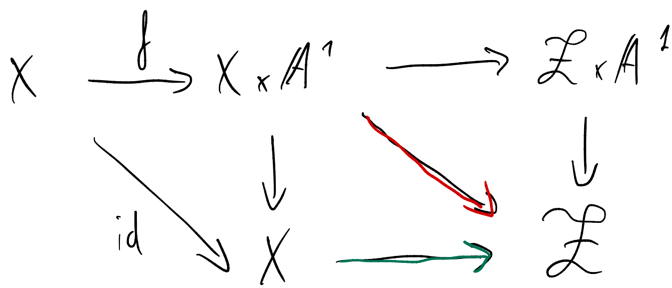
$X_0 \hookrightarrow X$, $X_1 \hookrightarrow X$ are $\mathbb{A}^1$-weak eq

in this case, X_0 and X_1 are $\mathbb{A}^1$-h-cobordant

$X \times \mathbb{A}^1 \to \mathbb{A}^1$ is an $\mathbb{A}^1$-h-c if X smooth and proper



$[X \times \mathbb{A}^1, \mathcal{Z}] \xrightarrow[\text{bij}]{f^*} [X, \mathcal{Z}]$ for $\mathcal{Z}$ $\mathbb{A}^1$-local



so $X \to X \times \mathbb{A}^1$ is $\mathbb{A}^1$-weak eq

$X \times \mathbb{A}^1 \to \mathbb{A}^1$ is an $\mathbb{A}^1$-h.c

$X \sim X$

$\mathbb{A}^1$-eq Classification of rational smooth proper surfaces

isomorphism classification: $\boxed{\mathbb{P}^2}$, $\boxed{Bl_{x_1}(\cdots(Bl_{x_n}(\mathbb{P}^2)))}$

$\boxed{\mathbb{F}_a}$, $\boxed{Bl_{y_1}(\cdots(Bl_{y_n}(\mathbb{F}_a)))}$

On $\mathbb{P}^1$: $\mathcal{O}(a)$ for $a \in \mathbb{Z}$ $\qquad a \geq 0$

$$\pi : \mathbb{P}(\underbrace{\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}(a)}_{M}) \to \mathbb{P}^1$$

$$U = \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$$

$$M|_U \simeq \mathcal{O}_U^{\boxed{2}} \qquad 2 - 1 = 1$$

$$\pi^{-1}(U) \simeq \mathbb{P}^1_U \simeq \mathbb{P}^1 \times \mathbb{A}^1 \simeq \mathbb{P}^2$$

$$\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$$

$$\mathcal{O}_{\mathbb{P}^1}(-a) \otimes (\mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(a)) \simeq \mathcal{O}_{\mathbb{P}^1}(b-a) \oplus \mathcal{O}_{\mathbb{P}^1}$$

$$\mathcal{O}_{\mathbb{P}^1}(a) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a))$$

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{O}_{\mathbb{P}^1}(a)$$

$$\underset{\substack{\| \\ C_a}}{\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a))} \hookrightarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a)) = \mathbb{F}_a$$

Fact 1

$x \in C_a(k)$ $\qquad \boxed{Bl_x(\mathbb{F}_a) \simeq Bl_{x'}(\mathbb{F}_{a-1})}$

with $x' \notin C_{a-1}(k)$

<u>Fact 2:</u> $Bl_x(\mathbb{P}^2) \simeq \mathbb{F}_1$.

<u>Theorem 3.2.1:</u> $k = \bar{k}$, S_n as the blow up of a finite collection of n arbitrary points of $\mathbb{P}^2$.

Any rational smooth proper surface is $\mathbb{A}^1$-weak eq to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or some S_n.

<u>Proof:</u> let's first show that we can move a blown up point.

Blowing up a moving point

Proposition 3.1.7: X smooth proper variety

$i : \mathbb{A}^2 \hookrightarrow X$ closed immersion

$\mathbb{A}^2 \searrow \mathbb{A}^n \nearrow$ open embedding

Γ = graph of i.

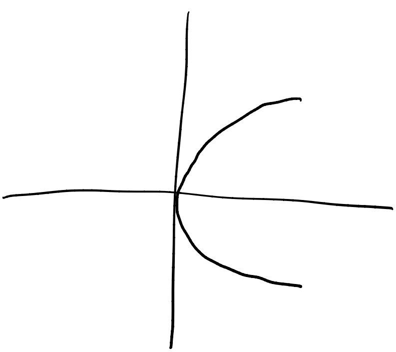
$\mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$

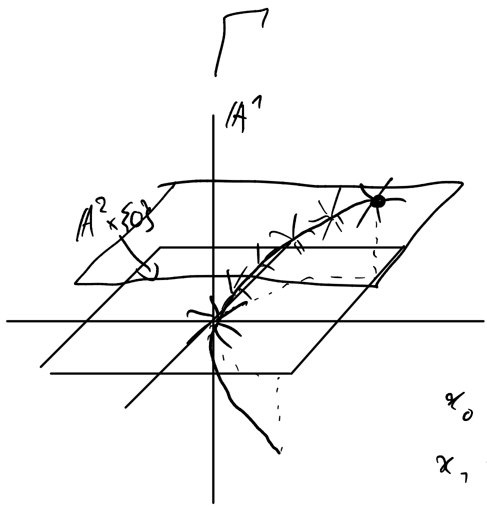
$\Gamma \hookrightarrow X \times \mathbb{A}^1$

the composite $Bl_\Gamma(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1 \to \mathbb{A}^1$ is an $\mathbb{A}^1$-h cobordism.

Drawing: for example $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$

$i : \mathbb{A}^1 \to \mathbb{A}^2$





$x_0 = i(0)$

$x_1 = i(1)$

$$Bl_{x_0}(X) \sim Bl_{x_1}(X)$$

Proposition: 3.2.7 <u>k is infinite</u> X is smooth proper k-variety * $f_1 : X_1 \to X$, $f_2 : X_2 \to X$ proper birational and composite of iterated blow ups of points. X_1 and X_2 are $\mathbb{A}^1$-eq iff $\mathrm{rk}\, \mathrm{Pic}\, X_1 = \mathrm{rk}\, \mathrm{Pic}\, X_2$, in this cas they are $\mathbb{A}^1$-h-cobordant

* and covered by affine spaces

Proof: notice that $\mathrm{Bl}_x X$ is still covered by affine spaces, still a smooth proper variety.

1. By 3.1.7, if x_1 and x_2 lies on a common $\mathbb{A}^1 \hookrightarrow X$ in X, then $\mathrm{Bl}_{x_1}(X) \sim_{\mathbb{A}^1\text{-}h\text{-}c} \mathrm{Bl}_{x_2}(X)$

1.bis: if x_1 and x_2 are not on a common line we take a chain (X covered by affine spaces $\Rightarrow X$ $\mathbb{A}^1$-chain connected), $x_1 = z_0 \sim z_1 \sim \cdots \sim z_n = x_2$

2. $\mathrm{Bl}_{x_1}(\mathrm{Bl}_{x_2}(X))$ we can always move x_1 out of the exceptional locus of $\mathrm{Bl}_{x_2}(X)$.

3. $\mathrm{Bl}_{x_1}(\mathrm{Bl}_{x_2}(X))$ with $x_1 \neq x_2$ in X ⟲
   $\mathrm{Bl}_{y_1}(\mathrm{Bl}_{y_2}(X))$ with $y_1 \neq y_2$ in X

$\sim$ stands for $\mathbb{A}^1$-h-cobordant

$$Bl_{x_1}(Bl_{x_2}(X)) \sim Bl_{y_1}(Bl_{x_2}(X))$$
$$= Bl_{\{y_1, x_2\}}(X)$$
$$\simeq Bl_{x_2}(Bl_{y_1}(X))$$
$$\sim Bl_{y_2}(Bl_{y_1}(X))$$
$$= Bl_{y_1}(Bl_{y_2}(X))$$

$$Bl_{x_1}(Bl_{x_2}(X)) \sim Bl_{y_1}(B_{y_2}(X))$$

4. $\Rightarrow Bl_{x_1}(\cdots(Bl_{x_n}(X)) \sim Bl_{y_1}(\cdots(Bl_{y_n}(X))$

5. $\left(X_1 \sim_{\mathbb{A}^1\text{-eq}} X_2\right) \Rightarrow \left(\mathrm{Pic}\, X_1 = \mathrm{Pic}\, X_2\right)$

if $\mathrm{rk}\,\mathrm{Pic}\, X_1 = \mathrm{rk}\,\mathrm{Pic}\, X_2$, then

the number of blow ups to get X_1 is the same as

____________________ X_2 because blowing

up a point adds a $\mathbb{Z}$-summand to the Picard group so $X_1 \sim_{\mathbb{A}^1\text{-h-cobordant}} X_2$ so $X_1 \sim_{\mathbb{A}^1\text{-eq}} X_2$.

Back to the proof of classification theorem

if X is rational smooth proper surface

$X = \mathbb{P}^2$ OK

$X = Bl_{x_1}(\cdots(Bl_{x_n}(\mathbb{P}^2)) \sim Bl_{x_1'}(\cdots(Bl_{x_n'}(\mathbb{P}^2))$ with $x_1', \dots, x_n' \in \mathbb{P}^2$

$X \sim S_n = Bl_{y_1}(\cdots(Bl_{y_n}(\mathbb{P}^2))$

$X = Bl_{x_1}(\cdots(Bl_{x_n}(\mathbb{F}_a))$ (*)

<u>Lemma</u>: $Bl_{x_1}(\cdots(Bl_{x_n}(\mathbb{F}_a)) \sim_{\mathbb{A}^1\text{-h-c}} Bl_{y_1}(\cdots(Bl_{y_n}(\mathbb{F}_{a-1}))$

$n \geq 1$, $a \geq 2$

Proof: $Bl_{x_n}(\mathbb{F}_a) \sim Bl_{x_n'}(\mathbb{F}_a)$ for $x_n' \in C_a(k)$
$\hookrightarrow \mathbb{F}_a$

and $\boxed{Bl_{x_n'}(\mathbb{F}_a) \cong Bl_{y_n'}(\mathbb{F}_{a-1})}$ for some $y_n' \in C_{a-1}(k)$

$Bl_{x_n}(\mathbb{F}_a) \sim Bl_{y_n'}(\mathbb{F}_{a-1}) \sim Bl_{y_n''}(\mathbb{F}_{a-1})$ $y_n'' \in C_{a-1}$

$$Bl_{x_1}(\dots(Bl_{x_{n'}}(\mathbb{F}_a)) \simeq Bl_{x_1}(\dots(Bl_{y_{n'}}(\mathbb{F}_{a-1}))$$

$$\wr\, \mathbb{A}^1\text{-}h. \qquad\qquad \wr\, \mathbb{A}^1\text{-}h$$

$$Bl_{x_1}(\dots(Bl_{x_n}(\mathbb{F}_a)) \qquad Bl_{y_1}(\dots(Bl_{y_n}(\mathbb{F}_{a-1}))$$

<u>Back to the proof</u>:

(*) if $X = Bl_{x_1}(\dots(Bl_{x_n}(\mathbb{F}_a))$

$$X \sim Bl_{y_1}(\dots(Bl_{y_n}(\mathbb{F}_1))$$

$$\wr\, Bl_x(\mathbb{P}^2)$$

$$X \sim S_{n+1}$$

$$\boxed{\mathbb{F}_a \underset{\mathbb{A}^1\text{-}h}{\sim} \mathbb{F}_b \quad \text{iff} \quad a \equiv b\ [2]}$$

if $X \simeq \mathbb{F}_a$

$X \sim \mathbb{F}_0$ or $\mathbb{F}_1$

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1 = Bl_x(\mathbb{P}^2)$